8 CHAPTER

Magnetic Forces, Materials, and Inductance

We are now ready to undertake the second half of the magnetic field problem, that of determining the forces and torques exerted by the magnetic field on other charges. The electric field causes a force to be exerted on a charge that may be either stationary or in motion; we will see that the steady magnetic field is capable of exerting a force only on a *moving* charge. This result appears reasonable; a magnetic field may be produced by moving charges and may exert forces on moving charges; a magnetic field cannot arise from stationary charges and cannot exert any force on a stationary charge.

This chapter initially considers the forces and torques on current-carrying conductors that may either be of a filamentary nature or possess a finite cross section with a known current density distribution. The problems associated with the motion of particles in a vacuum are largely avoided.

With an understanding of the fundamental effects produced by the magnetic field, we may then consider the varied types of magnetic materials, the analysis of elementary magnetic circuits, the forces on magnetic materials, and finally, the important electrical circuit concepts of self-inductance and mutual inductance. ■

8.1 FORCE ON A MOVING CHARGE

In an electric field, the definition of the electric field intensity shows us that the force on a charged particle is

$$\boxed{\mathbf{F} = Q\mathbf{E}} \tag{1}$$

The force is in the same direction as the electric field intensity (for a positive charge) and is directly proportional to both $\mathbf{E}$ and Q. If the charge is in motion, the force at any point in its trajectory is then given by (1).

A charged particle in motion in a magnetic field of flux density $\mathbf{B}$ is found experimentally to experience a force whose magnitude is proportional to the product of the magnitudes of the charge Q, its velocity $\mathbf{v}$, and the flux density $\mathbf{B}$, and to the sine of the angle between the vectors $\mathbf{v}$ and $\mathbf{B}$. The direction of the force is perpendicular to both $\mathbf{v}$ and $\mathbf{B}$ and is given by a unit vector in the direction of $\mathbf{v} \times \mathbf{B}$. The force may therefore be expressed as

$$\boxed{\mathbf{F} = Q\mathbf{v} \times \mathbf{B}} \tag{2}$$

A fundamental difference in the effect of the electric and magnetic fields on charged particles is now apparent, for a force which is always applied in a direction at right angles to the direction in which the particle is proceeding can never change the magnitude of the particle velocity. In other words, the *acceleration* vector is always normal to the velocity vector. The kinetic energy of the particle remains unchanged, and it follows that the steady magnetic field is incapable of transferring energy to the moving charge. The electric field, on the other hand, exerts a force on the particle which is independent of the direction in which the particle is progressing and therefore effects an energy transfer between field and particle in general.

The first two problems at the end of this chapter illustrate the different effects of electric and magnetic fields on the kinetic energy of a charged particle moving in free space.

The force on a moving particle arising from combined electric and magnetic fields is obtained easily by superposition,

$$\boxed{\mathbf{F} = Q(\mathbf{E} + \mathbf{v} \times \mathbf{B})} \tag{3}$$

This equation is known as the *Lorentz force equation,* and its solution is required in determining electron orbits in the magnetron, proton paths in the cyclotron, plasma characteristics in a magnetohydrodynamic (MHD) generator, or, in general, charged-particle motion in combined electric and magnetic fields.

D8.1. The point charge $Q = 18$ nC has a velocity of 5×10^6 m/s in the direction $\mathbf{a}_v = 0.60\mathbf{a}_x + 0.75\mathbf{a}_y + 0.30\mathbf{a}_z$. Calculate the magnitude of the force exerted on the charge by the field: (*a*) $\mathbf{B} = -3\mathbf{a}_x + 4\mathbf{a}_y + 6\mathbf{a}_z$ mT; (*b*) $\mathbf{E} = -3\mathbf{a}_x + 4\mathbf{a}_y + 6\mathbf{a}_z$ kV/m; (*c*) B and E acting together.

Ans. 660 μN; 140 μN; 670 μN

8.2 FORCE ON A DIFFERENTIAL CURRENT ELEMENT

The force on a charged particle moving through a steady magnetic field may be written as the differential force exerted on a differential element of charge,

$$d\mathbf{F} = dQ\,\mathbf{v} \times \mathbf{B} \tag{4}$$

Physically, the differential element of charge consists of a large number of very small, discrete charges occupying a volume which, although small, is much larger than the average separation between the charges. The differential force expressed by (4) is thus merely the sum of the forces on the individual charges. This sum, or resultant force, is not a force applied to a single object. In an analogous way, we might consider the differential gravitational force experienced by a small volume taken in a shower of falling sand. The small volume contains a large number of sand grains, and the differential force is the sum of the forces on the individual grains within the small volume.

If our charges are electrons in motion in a conductor, however, we can show that the force is transferred to the conductor and that the sum of this extremely large number of extremely small forces is of practical importance. Within the conductor, electrons are in motion throughout a region of immobile positive ions which form a crystalline array, giving the conductor its solid properties. A magnetic field which exerts forces on the electrons tends to cause them to shift position slightly and produces a small displacement between the centers of "gravity" of the positive and negative charges. The Coulomb forces between electrons and positive ions, however, tend to resist such a displacement. Any attempt to move the electrons, therefore, results in an attractive force between electrons and the positive ions of the crystalline lattice. The magnetic force is thus transferred to the crystalline lattice, or to the conductor itself. The Coulomb forces are so much greater than the magnetic forces in good conductors that the actual displacement of the electrons is almost immeasurable. The charge separation that does result, however, is disclosed by the presence of a slight potential difference across the conductor sample in a direction perpendicular to both the magnetic field and the velocity of the charges. The voltage is known as the *Hall voltage,* and the effect itself is called the *Hall effect*.

Figure 8.1 illustrates the direction of the Hall voltage for both positive and negative charges in motion. In Figure 8.1a, $\mathbf{v}$ is in the $-\mathbf{a}_x$ direction, $\mathbf{v} \times \mathbf{B}$ is in the $\mathbf{a}_y$ direction, and Q is positive, causing $\mathbf{F}_Q$ to be in the $\mathbf{a}_y$ direction; thus, the positive charges move to the right. In Figure 8.1b, $\mathbf{v}$ is now in the $+\mathbf{a}_x$ direction, $\mathbf{B}$ is still in the $\mathbf{a}_z$ direction, $\mathbf{v} \times \mathbf{B}$ is in the $-\mathbf{a}_y$ direction, and Q is negative; thus, $\mathbf{F}_Q$ is again in the $\mathbf{a}_y$ direction. Hence, the negative charges end up at the right edge. Equal currents provided by holes and electrons in semiconductors can therefore be differentiated by their Hall voltages. This is one method of determining whether a given semiconductor is n-type or p-type.

Devices employ the Hall effect to measure the magnetic flux density and, in some applications where the current through the device can be made proportional to the

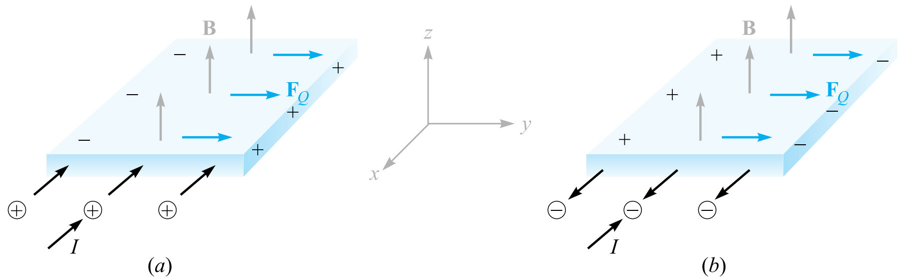

Figure 8.1 Equal currents directed into the material are provided by positive charges moving inward in (*a*) and negative charges moving outward in (*b*). The two cases can be distinguished by oppositely directed Hall voltages, as shown.

magnetic field across it, to serve as electronic wattmeters, squaring elements, and so forth.

Returning to (4), we may therefore say that if we are considering an element of moving charge in an electron beam, the force is merely the sum of the forces on the individual electrons in that small volume element, but if we are considering an element of moving charge within a conductor, the total force is applied to the solid conductor itself. We will now limit our attention to the forces on current-carrying conductors.

In Chapter 5 we defined convection current density in terms of the velocity of the volume charge density,

$$\boxed{\mathbf{J} = \rho_\nu \mathbf{v}}$$

The differential element of charge in (4) may also be expressed in terms of volume charge density,[1]

$$dQ = \rho_\nu d\nu$$

Thus

$$d\mathbf{F} = \rho_\nu d\nu\, \mathbf{v} \times \mathbf{B}$$

or

$$\boxed{d\mathbf{F} = \mathbf{J} \times \mathbf{B}\, d\nu} \qquad (5)$$

We saw in Chapter 7 that $\mathbf{J}\, d\nu$ may be interpreted as a differential current element; that is,

$$\mathbf{J}\, d\nu = \mathbf{K}\, dS = I\, d\mathbf{L}$$

[1]Remember that $d\nu$ is a differential volume element and not a differential increase in velocity.

and thus the Lorentz force equation may be applied to surface current density,

$$\boxed{d\mathbf{F} = \mathbf{K} \times \mathbf{B}\, dS} \tag{6}$$

or to a differential current filament,

$$\boxed{d\mathbf{F} = I\, d\mathbf{L} \times \mathbf{B}} \tag{7}$$

Integrating (5), (6), or (7) over a volume, a surface which may be either open or closed (why?), or a closed path, respectively, leads to the integral formulations

$$\mathbf{F} = \int_{\text{vol}} \mathbf{J} \times \mathbf{B}\, dv \tag{8}$$

$$\mathbf{F} = \int_S \mathbf{K} \times \mathbf{B}\, dS \tag{9}$$

and

$$\boxed{\mathbf{F} = \oint I\, d\mathbf{L} \times \mathbf{B} = -I \oint \mathbf{B} \times d\mathbf{L}} \tag{10}$$

One simple result is obtained by applying (7) or (10) to a straight conductor in a uniform magnetic field,

$$\boxed{\mathbf{F} = I\mathbf{L} \times \mathbf{B}} \tag{11}$$

The magnitude of the force is given by the familiar equation

$$F = BIL \sin\theta \tag{12}$$

where θ is the angle between the vectors representing the direction of the current flow and the direction of the magnetic flux density. Equation (11) or (12) applies only to a portion of the closed circuit, and the remainder of the circuit must be considered in any practical problem.

EXAMPLE 8.1

As a numerical example of these equations, consider Figure 8.2. We have a square loop of wire in the $z = 0$ plane carrying 2 mA in the field of an infinite filament on the y axis, as shown. We desire the total force on the loop.

Solution. The field produced in the plane of the loop by the straight filament is

$$\mathbf{H} = \frac{I}{2\pi x}\mathbf{a}_z = \frac{15}{2\pi x}\mathbf{a}_z \text{ A/m}$$

Therefore,

$$\mathbf{B} = \mu_0 \mathbf{H} = 4\pi \times 10^{-7}\mathbf{H} = \frac{3 \times 10^{-6}}{x}\mathbf{a}_z \text{ T}$$

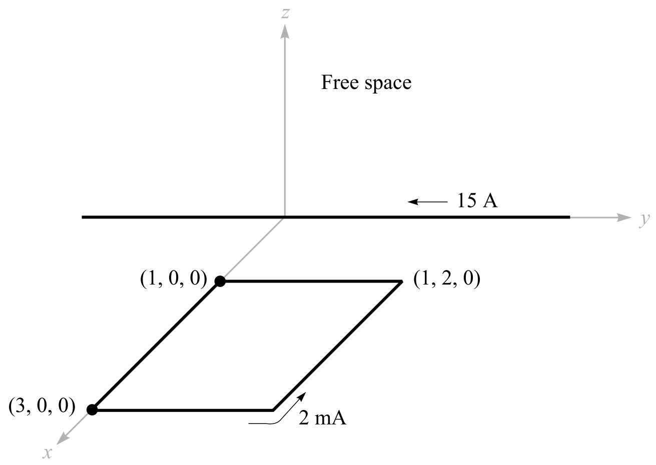

Figure 8.2 A square loop of wire in the xy plane carrying 2 mA is subjected to a nonuniform B field.

We use the integral form (10),

$$\mathbf{F} = -I \oint \mathbf{B} \times d\mathbf{L}$$

Let us assume a rigid loop so that the total force is the sum of the forces on the four sides. Beginning with the left side:

$$\mathbf{F} = -2 \times 10^{-3} \times 3 \times 10^{-6} \left[\int_{x=1}^{3} \frac{\mathbf{a}_z}{x} \times dx\, \mathbf{a}_x + \int_{y=0}^{2} \frac{\mathbf{a}_z}{3} \times dy\, \mathbf{a}_y \right.$$

$$\left. + \int_{x=3}^{1} \frac{\mathbf{a}_z}{x} \times dx\, \mathbf{a}_x + \int_{y=2}^{0} \frac{\mathbf{a}_z}{1} \times dy\, \mathbf{a}_y \right]$$

$$= -6 \times 10^{-9} \left[\ln x \Big|_1^3 \mathbf{a}_y + \frac{1}{3} y \Big|_0^2 (-\mathbf{a}_x) + \ln x \Big|_3^1 \mathbf{a}_y + y \Big|_2^0 (-\mathbf{a}_x) \right]$$

$$= -6 \times 10^{-9} \left[(\ln 3)\mathbf{a}_y - \frac{2}{3}\mathbf{a}_x + \left(\ln \frac{1}{3} \right) \mathbf{a}_y + 2\mathbf{a}_x \right]$$

$$= -8\mathbf{a}_x \text{ nN}$$

Thus, the net force on the loop is in the $-\mathbf{a}_x$ direction.

D8.2. The field $\mathbf{B} = -2\mathbf{a}_x + 3\mathbf{a}_y + 4\mathbf{a}_z$ mT is present in free space. Find the vector force exerted on a straight wire carrying 12 A in the $\mathbf{a}_{AB}$ direction, given $A(1, 1, 1)$ and: (*a*) $B(2, 1, 1)$; (*b*) $B(3, 5, 6)$.

Ans. $-48\mathbf{a}_y + 36\mathbf{a}_z$ mN; $12\mathbf{a}_x - 216\mathbf{a}_y + 168\mathbf{a}_z$ mN

D8.3. The semiconductor sample shown in Figure 8.1 is n-type silicon, having a rectangular cross section of 0.9 mm by 1.1 cm and a length of 1.3 cm. Assume the electron and hole mobilities are 0.13 and 0.03 $m^2/V \cdot s$, respectively, at the operating temperature. Let $B = 0.07$ T and the electric field intensity in the direction of the current flow be 800 V/m. Find the magnitude of: (*a*) the voltage across the sample length; (*b*) the drift velocity; (*c*) the transverse force per coulomb of moving charge caused by B; (*d*) the transverse electric field intensity; (*e*) the Hall voltage.

Ans. 10.40 V; 104.0 m/s; 7.28 N/C; 7.28 V/m; 80.1 mV

8.3 FORCE BETWEEN DIFFERENTIAL CURRENT ELEMENTS

The concept of the magnetic field was introduced to break into two parts the problem of finding the interaction of one current distribution on a second current distribution. It is possible to express the force on one current element directly in terms of a second current element without finding the magnetic field. Because we claimed that the magnetic-field concept simplifies our work, it then behooves us to show that avoidance of this intermediate step leads to more complicated expressions.

The magnetic field at point 2 due to a current element at point 1 was found to be

$$d\mathbf{H}_2 = \frac{I_1 d\mathbf{L}_1 \times \mathbf{a}_{R12}}{4\pi R_{12}^2}$$

Now, the differential force on a differential current element is

$$d\mathbf{F} = I\, d\mathbf{L} \times \mathbf{B}$$

and we apply this to our problem by letting $\mathbf{B}$ be $d\mathbf{B}_2$ (the differential flux density at point 2 caused by current element 1), by identifying $I\,d\mathbf{L}$ as $I_2 d\mathbf{L}_2$, and by symbolizing the differential amount of our differential force on element 2 as $d(d\mathbf{F}_2)$:

$$d(d\mathbf{F}_2) = I_2 d\mathbf{L}_2 \times d\mathbf{B}_2$$

Because $d\mathbf{B}_2 = \mu_0 d\mathbf{H}_2$, we obtain the force between two differential current elements,

$$d(d\mathbf{F}_2) = \mu_0 \frac{I_1 I_2}{4\pi R_{12}^2} d\mathbf{L}_2 \times (d\mathbf{L}_1 \times \mathbf{a}_{R12}) \tag{13}$$

EXAMPLE 8.2

As an example that illustrates the use (and misuse) of these results, consider the two differential current elements shown in Figure 8.3. We seek the differential force on $d\mathbf{L}_2$.

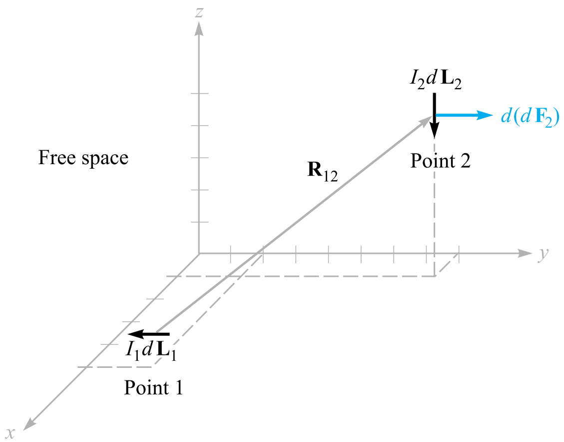

Figure 8.3 Given $P_1(5, 2, 1)$, $P_2(1, 8, 5)$, $I_1\, d\mathbf{L}_1 = -3\mathbf{a}_y$ A · m, and $I_2\, d\mathbf{L}_2 = -4\mathbf{a}_z$ A · m, the force on $I_2\, d\mathbf{L}_2$ is 8.56 nN in the $\mathbf{a}_y$ direction.

Solution. We have $I_1 d\mathbf{L}_1 = -3\mathbf{a}_y \text{A} \cdot \text{m}$ at $P_1(5, 2, 1)$, and $I_2 d\mathbf{L}_2 = -4\mathbf{a}_z \text{A} \cdot \text{m}$ at $P_2(1, 8, 5)$. Thus, $\mathbf{R}_{12} = -4\mathbf{a}_x + 6\mathbf{a}_y + 4\mathbf{a}_z$, and we may substitute these data into (13),

$$d(d\mathbf{F}_2) = \frac{4\pi 10^{-7}}{4\pi} \frac{(-4\mathbf{a}_z) \times [(-3\mathbf{a}_y) \times (-4\mathbf{a}_x + 6\mathbf{a}_y + 4\mathbf{a}_z)]}{(16 + 36 + 16)^{1.5}}$$
$$= 8.56\mathbf{a}_y \text{ nN}$$

Many chapters ago, when we discussed the force exerted by one point charge on another point charge, we found that the force on the first charge was the negative of that on the second. That is, the total force on the system was zero. This is not the case with the differential current elements, and $d(d\mathbf{F}_1) = -12.84\mathbf{a}_z$ nN in Example 8.2. The reason for this different behavior lies with the nonphysical nature of the current element. Whereas point charges may be approximated quite well by small charges, the continuity of current demands that a complete circuit be considered. This we shall now do.

The total force between two filamentary circuits is obtained by integrating twice:

$$\begin{aligned}\mathbf{F}_2 &= \mu_0 \frac{I_1 I_2}{4\pi} \oint \left[d\mathbf{L}_2 \times \oint \frac{d\mathbf{L}_1 \times \mathbf{a}_{R12}}{R_{12}^2} \right] \\ &= \mu_0 \frac{I_1 I_2}{4\pi} \oint \left[\oint \frac{\mathbf{a}_{R12} \times d\mathbf{L}_1}{R_{12}^2} \right] \times d\mathbf{L}_2\end{aligned} \tag{14}$$

Equation (14) is quite formidable, but the familiarity gained in Chapter 7 with the magnetic field should enable us to recognize the inner integral as the integral necessary to find the magnetic field at point 2 due to the current element at point 1.

Although we shall only give the result, it is not very difficult to use (14) to find the force of repulsion between two infinitely long, straight, parallel, filamentary

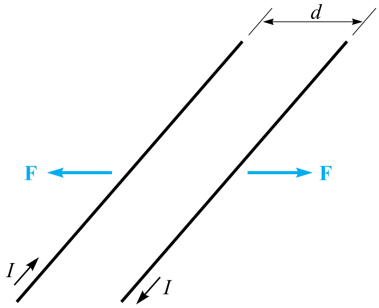

Figure 8.4 Two infinite parallel filaments with separation d and equal but opposite currents I experience a repulsive force of $\mu_0 I^2/(2\pi d)$ N/m.

conductors with separation d, and carrying equal but opposite currents I, as shown in Figure 8.4. The integrations are simple, and most errors are made in determining suitable expressions for $\mathbf{a}_{R12}$, $d\mathbf{L}_1$, and $d\mathbf{L}_2$. However, since the magnetic field intensity at either wire caused by the other is already known to be $I/(2\pi d)$, it is readily apparent that the answer is a force of $\mu_0 I^2/(2\pi d)$ newtons per meter length.

D8.4. Two differential current elements, $I_1\Delta\mathbf{L}_1 = 3 \times 10^{-6}\mathbf{a}_y$ A · m at $P_1(1, 0, 0)$ and $I_2\Delta\mathbf{L}_2 = 3\times 10^{-6}(-0.5\mathbf{a}_x + 0.4\mathbf{a}_y + 0.3\mathbf{a}_z)$ A · m at $P_2(2, 2, 2)$, are located in free space. Find the vector force exerted on: (*a*) $I_2\Delta\mathbf{L}_2$ by $I_1\Delta\mathbf{L}_1$; (*b*) $I_1\Delta\mathbf{L}_1$ by $I_2\Delta\mathbf{L}_2$.

Ans. $(-1.333\mathbf{a}_x + 0.333\mathbf{a}_y - 2.67\mathbf{a}_z)10^{-20}$ N; $(4.67\mathbf{a}_x + 0.667\mathbf{a}_z)10^{-20}$ N

8.4 FORCE AND TORQUE ON A CLOSED CIRCUIT

We have already obtained general expressions for the forces exerted on current systems. One special case is easily disposed of, for if we take our relationship for the force on a filamentary closed circuit, as given by Eq. (10), Section 8.2,

$$\mathbf{F} = -I \oint \mathbf{B} \times d\mathbf{L}$$

and assume a *uniform* magnetic flux density, then $\mathbf{B}$ may be removed from the integral:

$$\mathbf{F} = -I\mathbf{B} \times \oint d\mathbf{L}$$

However, we discovered during our investigation of closed line integrals in an electrostatic potential field that $\oint d\mathbf{L} = 0$, and therefore the force on a closed filamentary circuit in a uniform magnetic field is zero.

If the field is not uniform, the total force need not be zero.

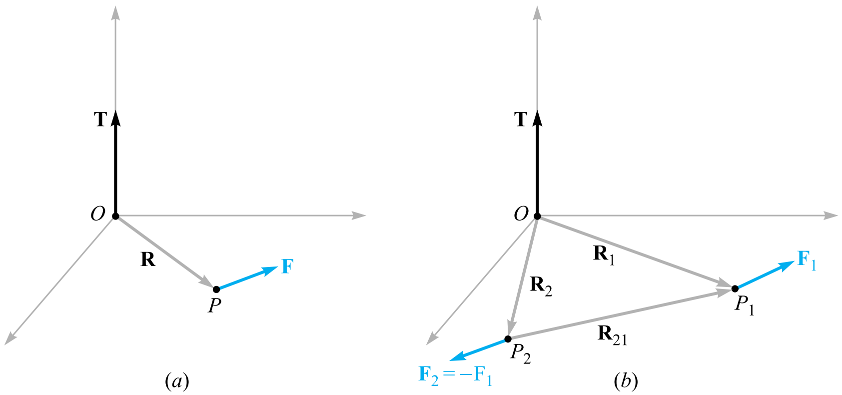

Figure 8.5 (*a*) Given a lever arm $\mathbf{R}$ extending from an origin O to a point P where force $\mathbf{F}$ is applied, the torque about O is $\mathbf{T} = \mathbf{R} \times \mathbf{F}$. (*b*) If $\mathbf{F}_2 = -\mathbf{F}_1$, then the torque $\mathbf{T} = \mathbf{R}_{21} \times \mathbf{F}_1$ is independent of the choice of origin for $\mathbf{R}_1$ and $\mathbf{R}_2$.

This result for uniform fields does not have to be restricted to filamentary circuits only. The circuit may contain surface currents or volume current density as well. If the total current is divided into filaments, the force on each one is zero, as we have shown, and the total force is again zero. Therefore, any real closed circuit carrying direct currents experiences a total vector force of zero in a uniform magnetic field.

Although the force is zero, the torque is generally not equal to zero.

In defining the *torque,* or *moment,* of a force, it is necessary to consider both an origin at or about which the torque is to be calculated, and the point at which the force is applied. In Figure 8.5*a*, we apply a force $\mathbf{F}$ at point P, and we establish an origin at O with a rigid lever arm $\mathbf{R}$ extending from O to P. The torque about point O is a vector whose magnitude is the product of the magnitudes of $\mathbf{R}$, of $\mathbf{F}$, and of the sine of the angle between these two vectors. The direction of the vector torque $\mathbf{T}$ is normal to both the force $\mathbf{F}$ and the lever arm $\mathbf{R}$ and is in the direction of progress of a right-handed screw as the lever arm is rotated into the force vector through the smaller angle. The torque is expressible as a cross product,

$$\mathbf{T} = \mathbf{R} \times \mathbf{F}$$

Now assume that two forces, $\mathbf{F}_1$ at P_1 and $\mathbf{F}_2$ at P_2, having lever arms $\mathbf{R}_1$ and $\mathbf{R}_2$ extending from a common origin O, as shown in Figure 8.5*b*, are applied to an object of fixed shape and that the object does not undergo any translation. Then the torque about the origin is

$$\mathbf{T} = \mathbf{R}_1 \times \mathbf{F}_1 + \mathbf{R}_2 \times \mathbf{F}_2$$

where

$$\mathbf{F}_1 + \mathbf{F}_2 = 0$$

and therefore

$$\mathbf{T} = (\mathbf{R}_1 - \mathbf{R}_2) \times \mathbf{F}_1 = \mathbf{R}_{21} \times \mathbf{F}_1$$

The vector $\mathbf{R}_{21} = \mathbf{R}_1 - \mathbf{R}_2$ joins the point of application of $\mathbf{F}_2$ to that of $\mathbf{F}_1$ and is independent of the choice of origin for the two vectors $\mathbf{R}_1$ and $\mathbf{R}_2$. Therefore, the torque is also independent of the choice of origin, provided that the total force is zero. This may be extended to any number of forces.

Consider the application of a vertically upward force at the end of a horizontal crank handle on an elderly automobile. This cannot be the only applied force, for if it were, the entire handle would be accelerated in an upward direction. A second force, equal in magnitude to that exerted at the end of the handle, is applied in a downward direction by the bearing surface at the axis of rotation. For a 40-N force on a crank handle 0.3 m in length, the torque is 12 N · m. This figure is obtained regardless of whether the origin is considered to be on the axis of rotation (leading to 12 N · m plus 0 N · m), at the midpoint of the handle (leading to 6 N · m plus 6 N · m), or at some point not even on the handle or an extension of the handle.

We may therefore choose the most convenient origin, and this is usually on the axis of rotation and in the plane containing the applied forces if the several forces are coplanar.

With this introduction to the concept of torque, let us now consider the torque on a differential current loop in a magnetic field $\mathbf{B}$. The loop lies in the xy plane (Figure 8.6); the sides of the loop are parallel to the x and y axes and are of length dx and dy. The value of the magnetic field at the center of the loop is taken as $\mathbf{B}_0$.

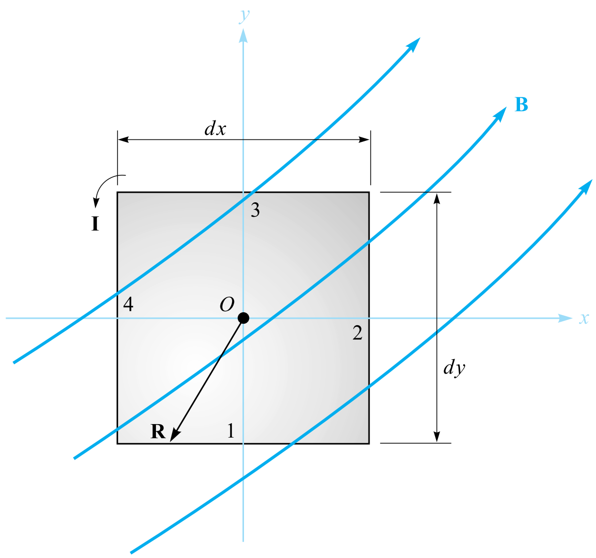

Figure 8.6 A differential current loop in a magnetic field $\mathbf{B}$. The torque on the loop is $d\mathbf{T} = I\,(dx\,dy\mathbf{a}_z) \times \mathbf{B}_0 = I\,d\mathbf{S} \times \mathbf{B}$.

Since the loop is of differential size, the value of **B** at all points on the loop may be taken as $\mathbf{B}_0$. (Why was this not possible in the discussion of curl and divergence?) The total force on the loop is therefore zero, and we are free to choose the origin for the torque at the center of the loop.

The vector force on side 1 is

$$d\mathbf{F}_1 = I\,dx\,\mathbf{a}_x \times \mathbf{B}_0$$

or

$$d\mathbf{F}_1 = I\,dx(B_{0y}\mathbf{a}_z - B_{0z}\mathbf{a}_y)$$

For this side of the loop the lever arm **R** extends from the origin to the midpoint of the side, $\mathbf{R}_1 = -\frac{1}{2}dy\,\mathbf{a}_y$, and the contribution to the total torque is

$$\begin{aligned} d\mathbf{T}_1 &= \mathbf{R}_1 \times d\mathbf{F}_1 \\ &= -\tfrac{1}{2}dy\,\mathbf{a}_y \times I\,dx(B_{0y}\mathbf{a}_z - B_{0z}\mathbf{a}_y) \\ &= -\tfrac{1}{2}dx\,dy\,I\,B_{0y}\mathbf{a}_x \end{aligned}$$

The torque contribution on side 3 is found to be the same,

$$\begin{aligned} d\mathbf{T}_3 &= \mathbf{R}_3 \times d\mathbf{F}_3 = \tfrac{1}{2}dy\,\mathbf{a}_y \times (-I\,dx\,\mathbf{a}_x \times \mathbf{B}_0) \\ &= -\tfrac{1}{2}dx\,dy\,IB_{0y}\mathbf{a}_x = d\mathbf{T}_1 \end{aligned}$$

and

$$d\mathbf{T}_1 + d\mathbf{T}_3 = -dx\,dy\,IB_{0y}\mathbf{a}_x$$

Evaluating the torque on sides 2 and 4, we find

$$d\mathbf{T}_2 + d\mathbf{T}_4 = dx\,dy\,IB_{0x}\mathbf{a}_y$$

and the total torque is then

$$d\mathbf{T} = I\,dx\,dy(B_{0x}\mathbf{a}_y - B_{0y}\mathbf{a}_x)$$

The quantity within the parentheses may be represented by a cross product,

$$d\mathbf{T} = I\,dx\,dy(\mathbf{a}_z \times \mathbf{B}_0)$$

or

$$\boxed{d\mathbf{T} = I\,d\mathbf{S} \times \mathbf{B}} \qquad (15)$$

where $d\mathbf{S}$ is the vector area of the differential current loop and the subscript on $\mathbf{B}_0$ has been dropped.

We now define the product of the loop current and the vector area of the loop as the differential *magnetic dipole moment* $d\mathbf{m}$, with units of $\mathrm{A} \cdot \mathrm{m}^2$. Thus

$$\boxed{d\mathbf{m} = I\,d\mathbf{S}} \qquad (16)$$

and

$$\boxed{d\mathbf{T} = d\mathbf{m} \times \mathbf{B}} \tag{17}$$

If we extend the results we obtained in Section 4.7 for the differential *electric* dipole by determining the torque produced on it by an *electric* field, we see a similar result,

$$d\mathbf{T} = d\mathbf{p} \times \mathbf{E}$$

Equations (15) and (17) are general results that hold for differential loops of any shape, not just rectangular ones. The torque on a circular or triangular loop is also given in terms of the vector surface or the moment by (15) or (17).

Because we selected a differential current loop so that we might assume **B** was constant throughout it, it follows that the torque on a *planar* loop of any size or shape in a *uniform* magnetic field is given by the same expression,

$$\boxed{\mathbf{T} = I\mathbf{S} \times \mathbf{B} = \mathbf{m} \times \mathbf{B}} \tag{18}$$

We should note that the torque on the current loop always tends to turn the loop so as to align the magnetic field produced by the loop with the applied magnetic field that is causing the torque. This is perhaps the easiest way to determine the direction of the torque.

EXAMPLE 8.3

To illustrate some force and torque calculations, consider the rectangular loop shown in Figure 8.7. Calculate the torque by using $\mathbf{T} = I\mathbf{S} \times \mathbf{B}$.

Solution. The loop has dimensions of 1 m by 2 m and lies in the uniform field $\mathbf{B}_0 = -0.6\mathbf{a}_y + 0.8\mathbf{a}_z$T. The loop current is 4 mA, a value that is sufficiently small to avoid causing any magnetic field that might affect $\mathbf{B}_0$.

We have

$$\mathbf{T} = 4 \times 10^{-3}[(1)(2)\mathbf{a}_z] \times (-0.6\mathbf{a}_y + 0.8\mathbf{a}_z) = 4.8\mathbf{a}_x \text{ mN} \cdot \text{m}$$

Thus, the loop tends to rotate about an axis parallel to the positive x axis. The small magnetic field produced by the 4 mA loop current tends to line up with $\mathbf{B}_0$.

EXAMPLE 8.4

Now let us find the torque once more, this time by calculating the total force and torque contribution for each side.

Solution. On side 1 we have

$$\begin{aligned} \mathbf{F}_1 &= I\mathbf{L}_1 \times \mathbf{B}_0 = 4 \times 10^{-3}(1\mathbf{a}_x) \times (-0.6\mathbf{a}_y + 0.8\mathbf{a}_z) \\ &= -3.2\mathbf{a}_y - 2.4\mathbf{a}_z \text{ mN} \end{aligned}$$

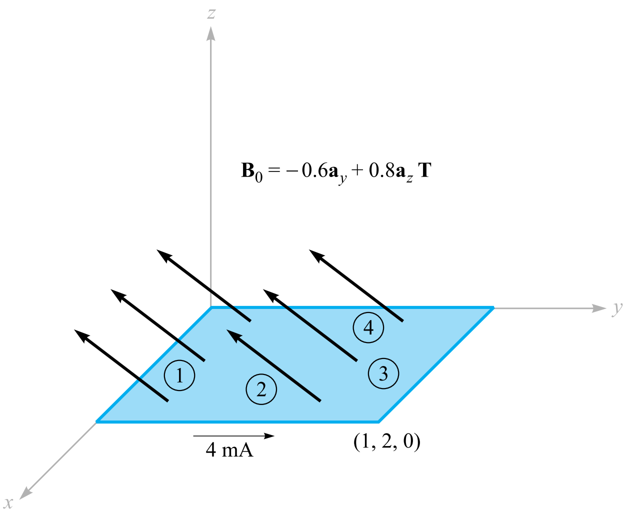

Figure 8.7 A rectangular loop is located in a uniform magnetic flux density $\mathbf{B}_0$.

On side 3 we obtain the negative of this result,

$$\mathbf{F}_3 = 3.2\mathbf{a}_y + 2.4\mathbf{a}_z \text{ mN}$$

Next, we attack side 2:

$$\begin{aligned}\mathbf{F}_2 &= I\mathbf{L}_2 \times \mathbf{B}_0 = 4 \times 10^{-3}(2\mathbf{a}_y) \times (-0.6\mathbf{a}_y + 0.8\mathbf{a}_z) \\ &= 6.4\mathbf{a}_x \text{ mN}\end{aligned}$$

with side 4 again providing the negative of this result,

$$\mathbf{F}_4 = -6.4\mathbf{a}_x \text{ mN}$$

Because these forces are distributed uniformly along each of the sides, we treat each force as if it were applied at the center of the side. The origin for the torque may be established anywhere since the sum of the forces is zero, and we choose the center of the loop. Thus,

$$\begin{aligned}\mathbf{T} &= \mathbf{T}_1 + \mathbf{T}_2 + \mathbf{T}_3 + \mathbf{T}_4 = \mathbf{R}_1 \times \mathbf{F}_1 + \mathbf{R}_2 \times \mathbf{F}_2 + \mathbf{R}_3 \times \mathbf{F}_3 + \mathbf{R}_4 \times \mathbf{F}_4 \\ &= (-1\mathbf{a}_y) \times (-3.2\mathbf{a}_y - 2.4\mathbf{a}_z) + (0.5\mathbf{a}_x) \times (6.4\mathbf{a}_x) \\ &\quad + (1\mathbf{a}_y) \times (3.2\mathbf{a}_y + 2.4\mathbf{a}_z) + (-0.5\mathbf{a}_x) \times (-6.4\mathbf{a}_x) \\ &= 2.4\mathbf{a}_x + 2.4\mathbf{a}_x = 4.8\mathbf{a}_x \text{ mN} \cdot \text{m}\end{aligned}$$

Crossing the loop moment with the magnetic flux density is certainly easier.

D8.5. A conducting filamentary triangle joins points $A(3, 1, 1)$, $B(5, 4, 2)$, and $C(1, 2, 4)$. The segment AB carries a current of 0.2 A in the $\mathbf{a}_{AB}$ direction. There is present a magnetic field $\mathbf{B} = 0.2\mathbf{a}_x - 0.1\mathbf{a}_y + 0.3\mathbf{a}_z$ T. Find: (a) the force on segment BC; (b) the force on the triangular loop; (c) the torque on the loop about an origin at A; (d) the torque on the loop about an origin at C.

Ans. $-0.08\mathbf{a}_x + 0.32\mathbf{a}_y + 0.16\mathbf{a}_z$ N; 0; $-0.16\mathbf{a}_x - 0.08\mathbf{a}_y + 0.08\mathbf{a}_z$ N · m; $-0.16\mathbf{a}_x - 0.08\mathbf{a}_y + 0.08\mathbf{a}_z$ N · m

8.5 THE NATURE OF MAGNETIC MATERIALS

We are now in a position to combine our knowledge of the action of a magnetic field on a current loop with a simple model of an atom and obtain some appreciation of the difference in behavior of various types of materials in magnetic fields.

Although accurate quantitative results can only be predicted through the use of quantum theory, the simple atomic model, which assumes that there is a central positive nucleus surrounded by electrons in various circular orbits, yields reasonable quantitative results and provides a satisfactory qualitative theory. An electron in an orbit is analogous to a small current loop (in which the current is directed oppositely to the direction of electron travel) and, as such, experiences a torque in an external magnetic field, the torque tending to align the magnetic field produced by the orbiting electron with the external magnetic field. If there were no other magnetic moments to consider, we would then conclude that all the orbiting electrons in the material would shift in such a way as to add their magnetic fields to the applied field, and thus that the resultant magnetic field at any point in the material would be greater than it would be at that point if the material were not present.

A second moment, however, is attributed to *electron spin*. Although it is tempting to model this phenomenon by considering the electron as spinning about its own axis and thus generating a magnetic dipole moment, satisfactory quantitative results are not obtained from such a theory. Instead, it is necessary to digest the mathematics of relativistic quantum theory to show that an electron may have a spin magnetic moment of about $\pm 9 \times 10^{-24}$ A · m^2; the plus and minus signs indicate that alignment aiding or opposing an external magnetic field is possible. In an atom with many electrons present, only the spins of those electrons in shells which are not completely filled will contribute to a magnetic moment for the atom.

A third contribution to the moment of an atom is caused by *nuclear spin*. Although this factor provides a negligible effect on the overall magnetic properties of materials, it is the basis of the nuclear magnetic resonance imaging (MRI) procedure provided by many of the larger hospitals.

Thus each atom contains many different component moments, and their combination determines the magnetic characteristics of the material and provides its general magnetic classification. We describe briefly six different types of material: diamagnetic, paramagnetic, ferromagnetic, antiferromagnetic, ferrimagnetic, and superparamagnetic.

Let us first consider atoms in which the small magnetic fields produced by the motion of the electrons in their orbits and those produced by the electron spin combine to produce a net field of zero. Note that we are considering here the fields produced by the electron motion itself in the absence of any external magnetic field; we might also describe this material as one in which the permanent magnetic moment $\mathbf{m}_0$ of each atom is zero. Such a material is termed *diamagnetic*. It would seem, therefore, that an external magnetic field would produce no torque on the atom, no realignment of the dipole fields, and consequently an internal magnetic field that is the same as the applied field. With an error that only amounts to about one part in a hundred thousand, this is correct.

Let us select an orbiting electron whose moment $\mathbf{m}$ is in the same direction as the applied field $\mathbf{B}_0$ (Figure 8.8). The magnetic field produces an outward force on the orbiting electron. Since the orbital radius is quantized and cannot change, the inward Coulomb force of attraction is also unchanged. The force unbalance created by the outward magnetic force must therefore be compensated for by a reduced orbital velocity. Hence, the orbital moment decreases, and a smaller internal field results.

If we had selected an atom for which $\mathbf{m}$ and $\mathbf{B}_0$ were opposed, the magnetic force would be inward, the velocity would increase, the orbital moment would increase, and greater cancellation of $\mathbf{B}_0$ would occur. Again a smaller internal field would result.

Metallic bismuth shows a greater diamagnetic effect than most other diamagnetic materials, among which are hydrogen, helium, the other "inert" gases, sodium chloride, copper, gold, silicon, germanium, graphite, and sulfur. We should also realize that the diamagnetic effect is present in all materials, because it arises from an interaction of the external magnetic field with every orbiting electron; however, it is overshadowed by other effects in the materials we shall consider next.

Now consider an atom in which the effects of the electron spin and orbital motion do not quite cancel. The atom as a whole has a small magnetic moment, but the random orientation of the atoms in a larger sample produces an *average* magnetic moment of zero. The material shows no magnetic effects in the absence of an external field.

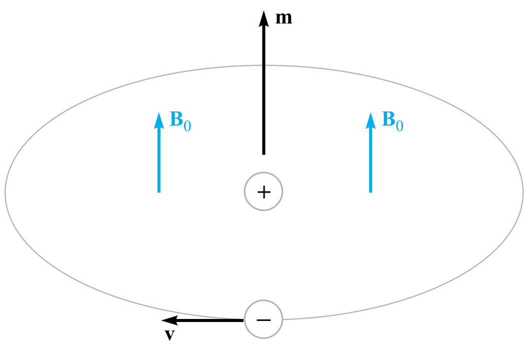

Figure 8.8 An orbiting electron is shown having a magnetic moment m in the same direction as an applied field B_0.

When an external field is applied, however, there is a small torque on each atomic moment, and these moments tend to become aligned with the external field. This alignment acts to increase the value of **B** within the material over the external value. However, the diamagnetic effect is still operating on the orbiting electrons and may counteract the increase. If the net result is a decrease in **B**, the material is still called diamagnetic. However, if there is an increase in **B**, the material is termed *paramagnetic*. Potassium, oxygen, tungsten, and the rare earth elements and many of their salts, such as erbium chloride, neodymium oxide, and yttrium oxide, one of the materials used in masers, are examples of paramagnetic substances.

The remaining four classes of material, ferromagnetic, antiferromagnetic, ferrimagnetic, and superparamagnetic, all have strong atomic moments. Moreover, the interaction of adjacent atoms causes an alignment of the magnetic moments of the atoms in either an aiding or exactly opposing manner.

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In *ferromagnetic* materials, each atom has a relatively large dipole moment, caused primarily by uncompensated electron spin moments. Interatomic forces cause these moments to line up in a parallel fashion over regions containing a large number of atoms. These regions are called *domains,* and they may have a variety of shapes and sizes ranging from one micrometer to several centimeters, depending on the size, shape, material, and magnetic history of the sample. Virgin ferromagnetic materials will have domains which each have a strong magnetic moment; the domain moments, however, vary in direction from domain to domain. The overall effect is therefore one of cancellation, and the material as a whole has no magnetic moment. Upon application of an external magnetic field, however, those domains which have moments in the direction of the applied field increase their size at the expense of their neighbors, and the internal magnetic field increases greatly over that of the external field alone. When the external field is removed, a completely random domain alignment is not usually attained, and a residual, or remnant, dipole field remains in the macroscopic structure. The fact that the magnetic moment of the material is different after the field has been removed, or that the magnetic state of the material is a function of its magnetic history, is called *hysteresis,* a subject which will be discussed again when magnetic circuits are studied in Section 8.8.

Ferromagnetic materials are not isotropic in single crystals, and we will therefore limit our discussion to polycrystalline materials, except for mentioning that one of the characteristics of anisotropic magnetic materials is magnetostriction, or the change in dimensions of the crystal when a magnetic field is impressed on it.

The only elements that are ferromagnetic at room temperature are iron, nickel, and cobalt, and they lose all their ferromagnetic characteristics above a temperature called the Curie temperature, which is 1043 K (770°C) for iron. Some alloys of these metals with each other and with other metals are also ferromagnetic, as for example alnico, an aluminum-nickel-cobalt alloy with a small amount of copper. At lower temperatures some of the rare earth elements, such as gadolinium and dysprosium, are ferromagnetic. It is also interesting that some alloys of nonferromagnetic metals are ferromagnetic, such as bismuth-manganese and copper-manganese-tin.

In *antiferromagnetic* materials, the forces between adjacent atoms cause the atomic moments to line up in an antiparallel fashion. The net magnetic moment is

Table 8.1 Characteristics of magnetic materials

Classification	Magnetic Moments	B Values	Comments
Diamagnetic	$\mathbf{m}_{orb} + \mathbf{m}_{spin} = 0$	$B_{int} < B_{appl}$	$B_{int} \doteq B_{appl}$
Paramagnetic	$\mathbf{m}_{orb} + \mathbf{m}_{spin} = \text{small}$	$B_{int} > B_{appl}$	$B_{int} \doteq B_{appl}$
Ferromagnetic	$\lvert\mathbf{m}_{spin}\rvert \gg \lvert\mathbf{m}_{orb}\rvert$	$B_{int} \gg B_{appl}$	Domains
Antiferromagnetic	$\lvert\mathbf{m}_{spin}\rvert \gg \lvert\mathbf{m}_{orb}\rvert$	$B_{int} \doteq B_{appl}$	Adjacent moments oppose
Ferrimagnetic	$\lvert\mathbf{m}_{spin}\rvert \gg \lvert\mathbf{m}_{orb}\rvert$	$B_{int} > B_{appl}$	Unequal adjacent moments oppose; low σ
Superparamagnetic	$\lvert\mathbf{m}_{spin}\rvert \gg \lvert\mathbf{m}_{orb}\rvert$	$B_{int} > B_{appl}$	Nonmagnetic matrix; recording tapes

zero, and antiferromagnetic materials are affected only slightly by the presence of an external magnetic field. This effect was first discovered in manganese oxide, but several hundred antiferromagnetic materials have been identified since then. Many oxides, sulfides, and chlorides are included, such as nickel oxide (NiO), ferrous sulfide (FeS), and cobalt chloride ($CoCl_2$). Antiferromagnetism is only present at relatively low temperatures, often well below room temperature. The effect is not of engineering importance at present.

The *ferrimagnetic* substances also show an antiparallel alignment of adjacent atomic moments, but the moments are not equal. A large response to an external magnetic field therefore occurs, although not as large as that in ferromagnetic materials. The most important group of ferrimagnetic materials are the *ferrites,* in which the conductivity is low, several orders of magnitude less than that of semiconductors. The fact that these substances have greater resistance than the ferromagnetic materials results in much smaller induced currents in the material when alternating fields are applied, as for example in transformer cores that operate at the higher frequencies. The reduced currents (eddy currents) lead to lower ohmic losses in the transformer core. The iron oxide magnetite (Fe_3O_4), a nickel-zinc ferrite ($Ni_{1/2}Zn_{1/2}Fe_2O_4$), and a nickel ferrite ($NiFe_2O_4$) are examples of this class of materials. Ferrimagnetism also disappears above the Curie temperature.

Superparamagnetic materials are composed of an assembly of ferromagnetic particles in a nonferromagnetic matrix. Although domains exist within the individual particles, the domain walls cannot penetrate the intervening matrix material to the adjacent particle. An important example is the magnetic tape used in audiotape or videotape recorders.

Table 8.1 summarizes the characteristics of the six types of magnetic materials we have discussed.

8.6 MAGNETIZATION AND PERMEABILITY

To place our description of magnetic materials on a more quantitative basis, we will now devote a page or so to showing how the magnetic dipoles act as a distributed source for the magnetic field. Our result will be an equation that looks very much like Ampère's circuital law, $\oint \mathbf{H} \cdot d\mathbf{L} = I$. The current, however, will be the movement of *bound* charges (orbital electrons, electron spin, and nuclear spin), and the field, which has the dimensions of **H**, will be called the magnetization **M**. The current produced by the bound charges is called a *bound current* or *Amperian current*.

Let us begin by defining the magnetization **M** in terms of the magnetic dipole moment **m**. The bound current I_b circulates about a path enclosing a differential area $d\mathbf{S}$, establishing a dipole moment ($\text{A} \cdot \text{m}^2$),

$$\mathbf{m} = I_b d\mathbf{S}$$

If there are n magnetic dipoles per unit volume and we consider a volume Δv, then the total magnetic dipole moment is found by the vector sum

$$\mathbf{m}_{\text{total}} = \sum_{i=1}^{n\Delta v} \mathbf{m}_i \tag{19}$$

Each of the $\mathbf{m}_i$ may be different. Next, we define the *magnetization* **M** as the *magnetic dipole moment per unit volume,*

$$\mathbf{M} = \lim_{\Delta v \to 0} \frac{1}{\Delta v} \sum_{i=1}^{n\Delta v} \mathbf{m}_i$$

and see that its units must be the same as for **H**, amperes per meter.

Now let us consider the effect of some alignment of the magnetic dipoles as the result of the application of a magnetic field. We shall investigate this alignment along a closed path, a short portion of which is shown in Figure 8.9. The figure shows several magnetic moments **m** that make an angle θ with the element of path $d\mathbf{L}$; each moment consists of a bound current I_b circulating about an area $d\mathbf{S}$. We are therefore considering a small volume, $dS \cos\theta dL$, or $d\mathbf{S} \cdot d\mathbf{L}$, within which there are $n d\mathbf{S} \cdot d\mathbf{L}$ magnetic dipoles. In changing from a random orientation to this partial alignment, the bound current crossing the surface enclosed by the path (to our left as we travel in the $\mathbf{a}_L$ direction in Figure 8.9) has increased by I_b for each of the $n d\mathbf{S} \cdot d\mathbf{L}$ dipoles. Thus the differential change in the net bound current I_B over the segment $d\mathbf{L}$ will be

$$dI_B = nI_b d\mathbf{S} \cdot d\mathbf{L} = \mathbf{M} \cdot d\mathbf{L} \tag{20}$$

and within an entire closed contour,

$$I_B = \oint \mathbf{M} \cdot d\mathbf{L} \tag{21}$$

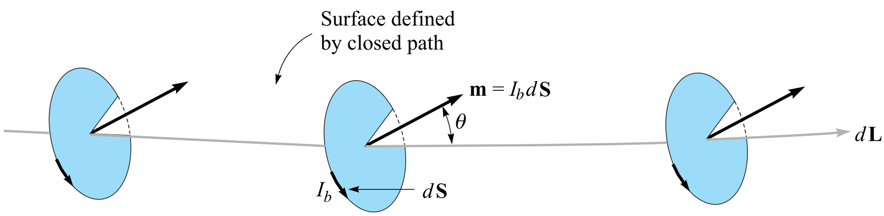

Figure 8.9 A section $d\mathbf{L}$ of a closed path along which magnetic dipoles have been partially aligned by some external magnetic field. The alignment has caused the bound current crossing the surface defined by the closed path to increase by $nI_b d\mathbf{S} \cdot d\mathbf{L}$ A.

Equation (21) merely says that if we go around a closed path and find dipole moments going our way more often than not, there will be a corresponding current composed of, for example, orbiting electrons crossing the interior surface.

This last expression has some resemblance to Ampère's circuital law, and we may now generalize the relationship between **B** and **H** so that it applies to media other than free space. Our present discussion is based on the forces and torques on differential current loops in a **B** field, and we therefore take **B** as our fundamental quantity and seek an improved definition of **H**. We thus write Ampère's circuital law in terms of the *total* current, bound plus free,

$$\oint \frac{\mathbf{B}}{\mu_0} \cdot d\mathbf{L} = I_T \tag{22}$$

where

$$I_T = I_{\mathbf{B}} + I$$

and I is the total *free* current enclosed by the closed path. Note that the free current appears without subscript since it is the most important type of current and will be the only current appearing in Maxwell's equations.

Combining these last three equations, we obtain an expression for the free current enclosed,

$$I = I_T - I_{\mathbf{B}} = \oint \left(\frac{\mathbf{B}}{\mu_0} - \mathbf{M} \right) \cdot d\mathbf{L} \tag{23}$$

We may now define **H** in terms of **B** and **M**,

$$\mathbf{H} = \frac{\mathbf{B}}{\mu_0} - \mathbf{M} \tag{24}$$

and we see that $\mathbf{B} = \mu_0 \mathbf{H}$ in free space where the magnetization is zero. This relationship is usually written in a form that avoids fractions and minus signs:

$$\boxed{\mathbf{B} = \mu_0(\mathbf{H} + \mathbf{M})} \tag{25}$$

We may now use our newly defined **H** field in (23),

$$I = \oint \mathbf{H} \cdot d\mathbf{L} \tag{26}$$

obtaining Ampère's circuital law in terms of the free currents.

Using the several current densities, we have

$$I_{\mathbf{B}} = \int_S \mathbf{J}_B \cdot d\mathbf{S}$$

$$I_T = \int_S \mathbf{J}_T \cdot d\mathbf{S}$$

$$I = \int_S \mathbf{J} \cdot d\mathbf{S}$$

With the help of Stokes' theorem, we may therefore transform (21), (26), and (22) into the equivalent curl relationships:

$$\nabla \times \mathbf{M} = \mathbf{J}_B$$

$$\nabla \times \frac{\mathbf{B}}{\mu_0} = \mathbf{J}_T$$

$$\boxed{\nabla \times \mathbf{H} = \mathbf{J}} \tag{27}$$

We will emphasize only (26) and (27), the two expressions involving the free charge, in the work that follows.

The relationship between **B**, **H**, and **M** expressed by (25) may be simplified for linear isotropic media where a magnetic susceptibility χ_m can be defined:

$$\boxed{\mathbf{M} = \chi_m \mathbf{H}} \tag{28}$$

Thus we have

$$\mathbf{B} = \mu_0(\mathbf{H} + \chi_m \mathbf{H})$$
$$= \mu_0 \mu_r \mathbf{H}$$

where

$$\mu_r = 1 + \chi_m \tag{29}$$

is defined as the *relative permeability* μ_r. We next define the *permeability* μ:

$$\mu = \mu_0 \mu_r \tag{30}$$

and this enables us to write the simple relationship between **B** and **H**,

$$\mathbf{B} = \mu \mathbf{H} \tag{31}$$

EXAMPLE 8.5

Given a ferrite material that we shall specify to be operating in a linear mode with $B = 0.05$ T, let us assume $\mu_r = 50$, and calculate values for χ_m, M, and H.

Solution. Because $\mu_r = 1 + \chi_m$, we have

$$\chi_m = \mu_r - 1 = 49$$

Also,

$$B = \mu_r \mu_0 H$$

and

$$H = \frac{0.05}{50 \times 4\pi \times 10^{-7}} = 796 \text{ A/m}$$

The magnetization is $M = \chi_m H$, or 39, 000 A/m. The alternate ways of relating B and H are, first,

$$B = \mu_0(H + M)$$

or

$$0.05 = 4\pi \times 10^{-7}(796 + 39,000)$$

showing that Amperian currents produce 49 times the magnetic field intensity that the free charges do; and second,

$$B = \mu_r \mu_0 H$$

or

$$0.05 = 50 \times 4\pi \times 10^{-7} \times 796$$

where we use a relative permeability of 50 and let this quantity account completely for the notion of the bound charges. We shall emphasize the latter interpretation in the chapters that follow.

The first two laws that we investigated for magnetic fields were the Biot-Savart law and Ampère's circuital law. Both were restricted to free space in their application. We may now extend their use to any homogeneous, linear, isotropic magnetic material that may be described in terms of a relative permeability μ_r.

Just as we found for anisotropic dielectric materials, the permeability of an anisotropic magnetic material must be given as a 3×3 matrix, and **B** and **H** are both 3×1 matrices. We have

$$B_x = \mu_{xx} H_x + \mu_{xy} H_y + \mu_{xz} H_z$$
$$B_y = \mu_{yx} H_x + \mu_{yy} H_y + \mu_{yz} H_z$$
$$B_z = \mu_{zx} H_x + \mu_{zy} H_y + \mu_{zz} H_z$$

For anisotropic materials, then, $\mathbf{B} = \mu\mathbf{H}$ is a matrix equation; however, $\mathbf{B} = \mu_0(\mathbf{H} + \mathbf{M})$ remains valid, although **B**, **H**, and **M** are no longer parallel in general. The most common anisotropic magnetic material is a single ferromagnetic crystal, although thin magnetic films also exhibit anisotropy. Most applications of ferromagnetic materials, however, involve polycrystalline arrays that are much easier to make.

Our definitions of susceptibility and permeability also depend on the assumption of linearity. Unfortunately, this is true only in the less interesting paramagnetic and diamagnetic materials for which the relative permeability rarely differs from unity by more than one part in a thousand. Some typical values of the susceptibility for diamagnetic materials are hydrogen, -2×10^{-5}; copper, -0.9×10^{-5}; germanium, -0.8×10^{-5}; silicon, -0.3×10^{-5}; and graphite, -12×10^{-5}. Several representative paramagnetic susceptibilities are oxygen, 2×10^{-6}; tungsten, 6.8×10^{-5}; ferric oxide (Fe_2O_3), 1.4×10^{-3}; and yttrium oxide (Y_2O_3), 0.53×10^{-6}. If we simply take the ratio of B to $\mu_0 H$ as the relative permeability of a ferromagnetic material, typical values of μ_r would range from 10 to 100, 000. Diamagnetic, paramagnetic, and antiferromagnetic materials are commonly said to be nonmagnetic.

D8.6. Find the magnetization in a magnetic material where: (*a*) $\mu = 1.8 \times 10^{-5}$ H/m and $H = 120$ A/m; (*b*) $\mu_r = 22$, there are 8.3×10^{28} atoms/m^3, and each atom has a dipole moment of 4.5×10^{-27} A $\cdot$ m^2; (*c*) $B = 300\,\mu$T and $\chi_m = 15$.

Ans. 1599 A/m; 374 A/m; 224 A/m

D8.7. The magnetization in a magnetic material for which $\chi_m = 8$ is given in a certain region as $150z^2\mathbf{a}_x$ A/m. At $z = 4$ cm, find the magnitude of: (*a*) $\mathbf{J}_T$; (*b*) J; (*c*) $\mathbf{J}_B$.

Ans. 13.5 A/m^2; 1.5 A/m^2; 12 A/m^2

8.7 MAGNETIC BOUNDARY CONDITIONS

We should have no difficulty in arriving at the proper boundary conditions to apply to **B**, **H**, and **M** at the interface between two different magnetic materials, for we have solved similar problems for both conducting materials and dielectrics. We need no new techniques.

Figure 8.10 shows a boundary between two isotropic homogeneous linear materials with permeabilities μ_1 and μ_2. The boundary condition on the normal components

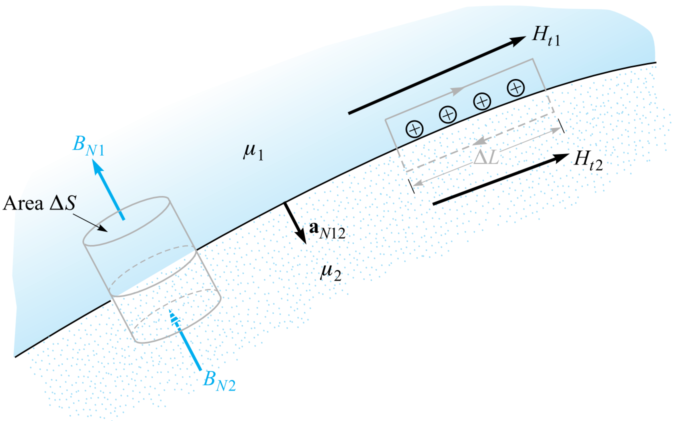

Figure 8.10 A gaussian surface and a closed path are constructed at the boundary between media 1 and 2, having permeabilities of μ_1 and μ_2, respectively. From this we determine the boundary conditions $B_{N1} = B_{N2}$ and $H_{t1} - H_{t2} = K$, the component of the surface current density directed into the page.

is determined by allowing the surface to cut a small cylindrical gaussian surface. Applying Gauss's law for the magnetic field from Section 7.5,

$$\oint_S \mathbf{B} \cdot d\mathbf{S} = 0$$

we find that

$$B_{N1}\Delta S - B_{N2}\Delta S = 0$$

or

$$\boxed{B_{N2} = B_{N1}} \tag{32}$$

Thus

$$H_{N2} = \frac{\mu_1}{\mu_2} H_{N1} \tag{33}$$

The normal component of **B** is continuous, but the normal component of **H** is discontinuous by the ratio μ_1/μ_2.

The relationship between the normal components of **M**, of course, is fixed once the relationship between the normal components of **H** is known. For linear magnetic materials, the result is written simply as

$$M_{N2} = \chi_{m2}\frac{\mu_1}{\mu_2} H_{N1} = \frac{\chi_{m2}\mu_1}{\chi_{m1}\mu_2} M_{N1} \tag{34}$$

Next, Ampère's circuital law

$$\oint \mathbf{H} \cdot d\mathbf{L} = I$$

is applied about a small closed path in a plane normal to the boundary surface, as shown to the right in Figure 8.10. Taking a clockwise trip around the path, we find that

$$H_{t1}\Delta L - H_{t2}\Delta L = K\Delta L$$

where we assume that the boundary may carry a surface current **K** whose component normal to the plane of the closed path is K. Thus

$$\boxed{H_{t1} - H_{t2} = K} \tag{35}$$

The directions are specified more exactly by using the cross product to identify the tangential components,

$$(\mathbf{H}_1 - \mathbf{H}_2) \times \mathbf{a}_{N12} = \mathbf{K}$$

where $\mathbf{a}_{N12}$ is the unit normal at the boundary directed from region 1 to region 2. An equivalent formulation in terms of the vector tangential components may be more convenient for **H**:

$$\mathbf{H}_{t1} - \mathbf{H}_{t2} = \mathbf{a}_{N12} \times \mathbf{K}$$

For tangential **B**, we have

$$\frac{B_{t1}}{\mu_1} - \frac{B_{t2}}{\mu_2} = K \tag{36}$$

The boundary condition on the tangential component of the magnetization for linear materials is therefore

$$M_{t2} = \frac{\chi_{m2}}{\chi_{m1}} M_{t1} - \chi_{m2} K \tag{37}$$

The last three boundary conditions on the tangential components are much simpler, of course, if the surface current density is zero. This is a free current density, and it must be zero if neither material is a conductor.

EXAMPLE 8.6

To illustrate these relationships with an example, let us assume that $\mu = \mu_1 = 4\ \mu$H/m in region 1 where $z > 0$, whereas $\mu_2 = 7\ \mu$H/m in region 2 wherever $z < 0$. Moreover, let $\mathbf{K} = 80\mathbf{a}_x$ A/m on the surface $z = 0$. We establish a field, $\mathbf{B}_1 = 2\mathbf{a}_x - 3\mathbf{a}_y + \mathbf{a}_z$ mT, in region 1 and seek the value of $\mathbf{B}_2$.

Solution. The normal component of $\mathbf{B}_1$ is

$$\mathbf{B}_{N1} = (\mathbf{B}_1 \cdot \mathbf{a}_{N12})\mathbf{a}_{N12} = [(2\mathbf{a}_x - 3\mathbf{a}_y + \mathbf{a}_z) \cdot (-\mathbf{a}_z)](-\mathbf{a}_z) = \mathbf{a}_z \text{ mT}$$

Thus,

$$\mathbf{B}_{N2} = \mathbf{B}_{N1} = \mathbf{a}_z \text{ mT}$$

We next determine the tangential components:

$$\mathbf{B}_{t1} = \mathbf{B}_1 - \mathbf{B}_{N1} = 2\mathbf{a}_x - 3\mathbf{a}y \text{ mT}$$

and

$$\mathbf{H}_{t1} = \frac{\mathbf{B}_{t1}}{\mu_1} = \frac{(2\mathbf{a}_x - 3\mathbf{a}_y)10^{-3}}{4 \times 10^{-6}} = 500\mathbf{a}_x - 750\mathbf{a}_y \text{ A/m}$$

Thus,

$$\begin{aligned}\mathbf{H}_{t2} = \mathbf{H}_{t1} - \mathbf{a}_{N12} \times \mathbf{K} &= 500\mathbf{a}_x - 750\mathbf{a}_y - (-\mathbf{a}_z) \times 80\mathbf{a}_x \\ &= 500\mathbf{a}_x - 750\mathbf{a}_y + 80\mathbf{a}_y = 500\mathbf{a}_x - 670\mathbf{a}_y \text{ A/m}\end{aligned}$$

and

$$\mathbf{B}_{t2} = \mu_2 \mathbf{H}_{t2} = 7 \times 10^{-6}(500\mathbf{a}_x - 670\mathbf{a}_y) = 3.5\mathbf{a}_x - 4.69\mathbf{a}_y \text{ mT}$$

Therefore,

$$\mathbf{B}_2 = \mathbf{B}_{N2} + \mathbf{B}_{t2} = 3.5\mathbf{a}_x - 4.69\mathbf{a}_y + \mathbf{a}_z \text{ mT}$$

D8.8. Let the permittivity be 5 μH/m in region A where $x < 0$, and 20 μH/m in region B where $x > 0$. If there is a surface current density $\mathbf{K} = 150\mathbf{a}_y - 200\mathbf{a}_z$ A/m at $x = 0$, and if $H_A = 300\mathbf{a}_x - 400\mathbf{a}_y + 500\mathbf{a}_z$ A/m, find: (a) $|\mathbf{H}_{tA}|$; (b) $|\mathbf{H}_{NA}|$; (c) $|\mathbf{H}_{tB}|$; (d) $|\mathbf{H}_{NB}|$.

Ans. 640 A/m; 300 A/m; 695 A/m; 75 A/m

8.8 THE MAGNETIC CIRCUIT

In this section, we digress briefly to discuss the fundamental techniques involved in solving a class of magnetic problems known as magnetic circuits. As we will see shortly, the name arises from the great similarity to the dc-resistive-circuit analysis with which it is assumed we are all familiar. The only important difference lies in the nonlinear nature of the ferromagnetic portions of the magnetic circuit; the methods which must be adopted are similar to those required in nonlinear electric circuits which contain diodes, thermistors, incandescent filaments, and other nonlinear elements.

As a convenient starting point, let us identify those field equations on which resistive circuit analysis is based. At the same time we will point out or derive the analogous equations for the magnetic circuit. We begin with the electrostatic potential and its relationship to electric field intensity,

$$\mathbf{E} = -\nabla V \tag{38a}$$

The scalar magnetic potential has already been defined, and its analogous relation to the magnetic field intensity is

$$\boxed{\mathbf{H} = -\nabla V_m} \tag{38b}$$

In dealing with magnetic circuits, it is convenient to call V_m the *magnetomotive force,* or mmf, and we shall acknowledge the analogy to the electromotive force, or emf, by doing so. The units of the mmf are, of course, amperes, but it is customary to recognize that coils with many turns are often employed by using the term "ampere-turns." Remember that no current may flow in any region in which V_m is defined.

The electric potential difference between points A and B may be written as

$$V_{AB} = \int_A^B \mathbf{E} \cdot d\mathbf{L} \tag{39a}$$

and the corresponding relationship between the mmf and the magnetic field intensity,

$$\boxed{V_{mAB} = \int_A^B \mathbf{H} \cdot d\mathbf{L}} \tag{39b}$$

was developed in Chapter 7, where we learned that the path selected must not cross the chosen barrier surface.

Ohm's law for the electric circuit has the point form

$$\mathbf{J} = \sigma \mathbf{E} \tag{40a}$$

and we see that the magnetic flux density will be the analog of the current density,

$$\boxed{\mathbf{B} = \mu \mathbf{H}} \tag{40b}$$

To find the total current, we must integrate:

$$I = \int_S \mathbf{J} \cdot d\mathbf{S} \tag{41a}$$

A corresponding operation is necessary to determine the total magnetic flux flowing through the cross section of a magnetic circuit:

$$\boxed{\Phi = \int_S \mathbf{B} \cdot d\mathbf{S}} \tag{41b}$$

We then defined resistance as the ratio of potential difference and current, or

$$V = IR \tag{42a}$$

and we shall now define *reluctance* as the ratio of the magnetomotive force to the total flux; thus

$$\boxed{V_m = \Phi \Re} \tag{42b}$$

where reluctance is measured in ampere-turns per weber ($\mathrm{A \cdot t/Wb}$). In resistors that are made of a linear isotropic homogeneous material of conductivity σ and have a uniform cross section of area S and length d, the total resistance is

$$R = \frac{d}{\sigma S} \tag{43a}$$

If we are fortunate enough to have such a linear isotropic homogeneous magnetic material of length d and uniform cross section S, then the total reluctance is

$$\boxed{\Re = \frac{d}{\mu S}} \tag{43b}$$

The only such material to which we shall commonly apply this relationship is air.

Finally, let us consider the analog of the source voltage in an electric circuit. We know that the closed line integral of $\mathbf{E}$ is zero,

$$\oint \mathbf{E} \cdot d\mathbf{L} = 0$$

In other words, Kirchhoff's voltage law states that the rise in potential through the source is exactly equal to the fall in potential through the load. The expression for magnetic phenomena takes on a slightly different form,

$$\oint \mathbf{H} \cdot d\mathbf{L} = I_{\text{total}}$$

for the closed line integral is not zero. Because the total current linked by the path is usually obtained by allowing a current I to flow through an N-turn coil, we may express this result as

$$\boxed{\oint \mathbf{H} \cdot d\mathbf{L} = NI} \tag{44}$$

In an electric circuit, the voltage source is a part of the closed path; in the magnetic circuit, the current-carrying coil will surround or link the magnetic circuit. In tracing a magnetic circuit, we will not be able to identify a pair of terminals at which the magnetomotive force is applied. The analogy is closer here to a pair of coupled circuits in which induced voltages exist (and in which we will see in Chapter 9 that the closed line integral of $\mathbf{E}$ is also not zero).

Let us try out some of these ideas on a simple magnetic circuit. In order to avoid the complications of ferromagnetic materials at this time, we will assume that we have an air-core toroid with 500 turns, a cross-sectional area of 6 cm^2, a mean radius of 15 cm, and a coil current of 4 A. As we already know, the magnetic field is confined to the interior of the toroid, and if we consider the closed path of our magnetic circuit along the mean radius, we link 2000 A · t,

$$V_{m,\text{ source}} = 2000 \text{ A} \cdot \text{t}$$

Although the field in the toroid is not quite uniform, we may assume that it is, for all practical purposes, and calculate the total reluctance of the circuit as

$$\Re = \frac{d}{\mu S} = \frac{2\pi(0.15)}{4\pi 10^{-7} \times 6 \times 10^{-4}} = 1.25 \times 10^{9} \text{ A}\cdot\text{t/Wb}$$

Thus

$$\Phi = \frac{V_{m,S}}{\Re} = \frac{2000}{1.25 \times 10^{9}} = 1.6 \times 10^{-6} \text{ Wb}$$

This value of the total flux is in error by less than $\frac{1}{4}$ percent, in comparison with the value obtained when the exact distribution of flux over the cross section is used.

Hence

$$B = \frac{\Phi}{S} = \frac{1.6 \times 10^{-6}}{6 \times 10^{-4}} = 2.67 \times 10^{-3} \text{ T}$$

and finally,

$$H = \frac{B}{\mu} = \frac{2.67 \times 10^{-3}}{4\pi 10^{-7}} = 2120 \text{ A}\cdot\text{t/m}$$

As a check, we may apply Ampère's circuital law directly in this symmetrical problem,

$$H_{\phi} 2\pi r = NI$$

and obtain

$$H_\phi = \frac{NI}{2\pi r} = \frac{500 \times 4}{6.28 \times 0.15} = 2120 \text{ A/m}$$

at the mean radius.

Our magnetic circuit in this example does not give us any opportunity to find the mmf across different elements in the circuit, for there is only one type of material. The analogous electric circuit is, of course, a single source and a single resistor. We could make it look just as long as the preceding analysis, however, if we found the current density, the electric field intensity, the total current, the resistance, and the source voltage.

More interesting and more practical problems arise when ferromagnetic materials are present in the circuit. Let us begin by considering the relationship between B and H in such a material. We may assume that we are establishing a curve of B versus H for a sample of ferromagnetic material which is completely demagnetized; both B and H are zero. As we begin to apply an mmf, the flux density also rises, but not linearly, as the experimental data of Figure 8.11 show near the origin. After H reaches a value of about 100 $\text{A} \cdot \text{t/m}$, the flux density rises more slowly and begins to saturate when H is several hundred $\text{A} \cdot \text{t/m}$. Having reached partial saturation, let us now turn to Figure 8.12, where we may continue our experiment at point x by reducing H. As we do so, the effects of hysteresis begin to show, and we do not retrace our original curve. Even after H is zero, $B = B_r$, the remnant flux density. As H is reversed, then brought back to zero, and the complete cycle traced several times, the hysteresis loop of Figure 8.12 is obtained. The mmf required to reduce the flux density to zero is identified as H_c, the coercive "force." For smaller maximum values of H, smaller

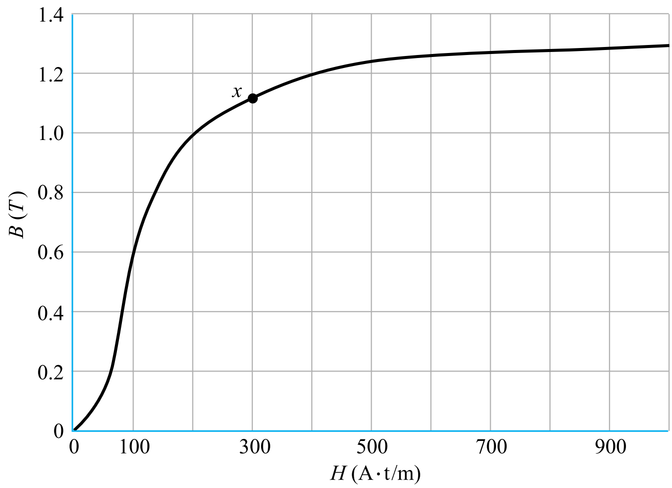

Figure 8.11 Magnetization curve of a sample of silicon sheet steel.

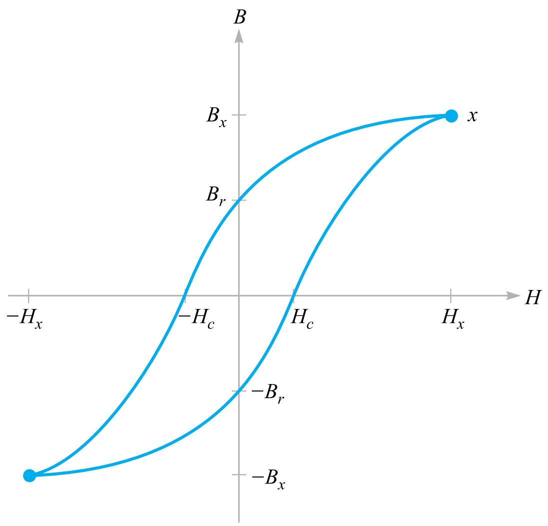

Figure 8.12 A hysteresis loop for silicon steel. The coercive force H_c and remnant flux density B_r are indicated.

hysteresis loops are obtained, and the locus of the tips is about the same as the virgin magnetization curve of Figure 8.11.

EXAMPLE 8.7

Let us use the magnetization curve for silicon steel to solve a magnetic circuit problem that is slightly different from our previous example. We use a steel core in the toroid, except for an air gap of 2 mm. Magnetic circuits with air gaps occur because gaps are deliberately introduced in some devices, such as inductors, which must carry large direct currents, because they are unavoidable in other devices such as rotating machines, or because of unavoidable problems in assembly. There are still 500 turns about the toroid, and we ask what current is required to establish a flux density of 1 T everywhere in the core.

Solution. This magnetic circuit is analogous to an electric circuit containing a voltage source and two resistors, one of which is nonlinear. Because we are given the "current," it is easy to find the "voltage" across each series element, and hence the total "emf." In the air gap,

$$\Re_{\text{air}} = \frac{d_{\text{air}}}{\mu S} = \frac{2 \times 10^{-3}}{4\pi 10^{-7} \times 6 \times 10^{-4}} = 2.65 \times 10^{6} \text{ A}\cdot\text{t/Wb}$$

Knowing the total flux,

$$\Phi = BS = 1(6 \times 10^{-4}) = 6 \times 10^{-4} \text{ Wb}$$

which is the same in both steel and air, we may find the mmf required for the gap,

$$V_{m,\text{air}} = (6 \times 10^{-4})(2.65 \times 10^{6}) = 1590 \text{ A}\cdot\text{t}$$

Referring to Figure 8.11, a magnetic field strength of 200 A · t/m is required to produce a flux density of 1 T in the steel. Thus,

$$\begin{aligned} H_{\text{steel}} &= 200 \text{ A}\cdot\text{t} \\ V_{m,\text{steel}} &= H_{\text{steel}} d_{\text{steel}} = 200 \times 0.30\pi \\ &= 188 \text{ A}\cdot\text{t} \end{aligned}$$

The total mmf is therefore 1778 A·t, and a coil current of 3.56 A is required.

We have made several approximations in obtaining this answer. We have already mentioned the lack of a completely uniform cross section, or cylindrical symmetry; the path of every flux line is not of the same length. The choice of a "mean" path length can help compensate for this error in problems in which it may be more important than it is in our example. Fringing flux in the air gap is another source of error, and formulas are available by which we may calculate an effective length and cross-sectional area for the gap which will yield more accurate results. There is also a leakage flux between the turns of wire, and in devices containing coils concentrated on one section of the core, a few flux lines bridge the interior of the toroid. Fringing and leakage are problems that seldom arise in the electric circuit because the ratio of the conductivities of air and the conductive or resistive materials used is so high. In contrast, the magnetization curve for silicon steel shows that the ratio of H to B in the steel is about 200 up to the "knee" of the magnetization curve; this compares with a ratio in air of about 800, 000. Thus, although flux prefers steel to air by the commanding ratio of 4000 to 1, this is not very close to the ratio of conductivities of, say, 10^{15} for a good conductor and a fair insulator.

EXAMPLE 8.8

As a last example, let us consider the reverse problem. Given a coil current of 4 A in the magnetic circuit of Example 8.7, what will the flux density be?

Solution. First let us try to linearize the magnetization curve by a straight line from the origin to $B = 1$, $H = 200$. We then have $B = H/200$ in steel and $B = \mu_0 H$ in air. The two reluctances are found to be 0.314×10^6 for the steel path and 2.65×10^6 for the air gap, or 2.96×10^6 A · t/Wb total. Since V_m is 2000 A · t, the flux is 6.76×10^{-4} Wb, and $B = 1.13$ T. A more accurate solution may be obtained by assuming several values of B and calculating the necessary mmf. Plotting the results enables us to determine the true value of B by interpolation. With this method we obtain $B = 1.10$ T. The good accuracy of the linear model results from the fact that the reluctance of the air gap in a magnetic circuit is often much greater than the reluctance of the ferromagnetic portion of the circuit. A relatively poor approximation for the iron or steel can thus be tolerated.

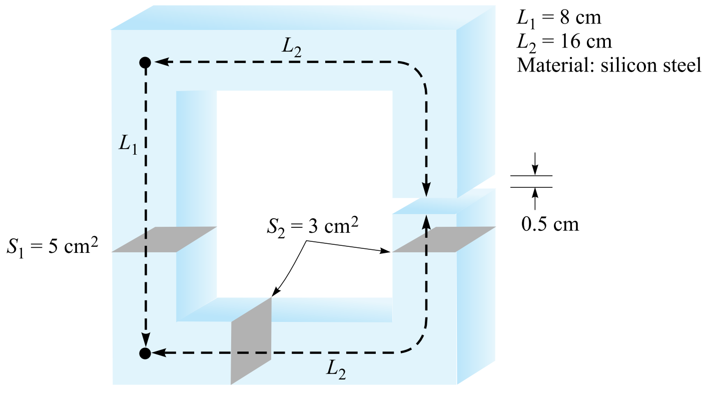

Figure 8.13 See Problem D8.9.

D8.9. Given the magnetic circuit of Figure 8.13, assume $B = 0.6$ T at the midpoint of the left leg and find: (*a*) $V_{m,\text{air}}$; (*b*) $V_{m,\text{steel}}$; (*c*) the current required in a 1300-turn coil linking the left leg.

Ans. 3980 A · t; 72 A · t; 3.12 A

D8.10. The magnetization curve for material X under normal operating conditions may be approximated by the expression $B = (H/160)(0.25 + e^{-H/320})$, where H is in A/m and B is in T. If a magnetic circuit contains a 12 cm length of material X, as well as a 0.25-mm air gap, assume a uniform cross section of 2.5 cm^2 and find the total mmf required to produce a flux of (*a*) 10 μWb; (*b*) 100 μWb.

Ans. 8.58 A · t; 86.7 A · t

8.9 POTENTIAL ENERGY AND FORCES ON MAGNETIC MATERIALS

In the electrostatic field we first introduced the point charge and the experimental law of force between point charges. After defining electric field intensity, electric flux density, and electric potential, we were able to find an expression for the energy in an electrostatic field by establishing the work necessary to bring the prerequisite point charges from infinity to their final resting places. The general expression for energy is

$$W_E = \frac{1}{2}\int_{\text{vol}} \mathbf{D}\cdot\mathbf{E}\,dv \tag{45}$$

where a linear relationship between **D** and **E** is assumed.

This is not as easily done for the steady magnetic field. It would seem that we might assume two simple sources, perhaps two current sheets, find the force on one due to the other, move the sheet a differential distance against this force, and equate the necessary work to the change in energy. If we did, we would be wrong, because Faraday's law (coming up in Chapter 9) shows that there will be a voltage induced in the moving current sheet against which the current must be maintained. Whatever source is supplying the current sheet turns out to receive half the energy we are putting into the circuit by moving it.

In other words, energy density in the magnetic field may be determined more easily after time-varying fields are discussed. We will develop the appropriate expression in discussing Poynting's theorem in Chapter 11.

An alternate approach would be possible at this time, however, for we might define a magnetostatic field based on assumed magnetic poles (or "magnetic charges"). Using the scalar magnetic potential, we could then develop an energy expression by methods similar to those used in obtaining the electrostatic energy relationship. These new magnetostatic quantities we would have to introduce would be too great a price to pay for one simple result, and we will therefore merely present the result at this time and show that the same expression arises in the Poynting theorem later. The total energy stored in a steady magnetic field in which **B** is linearly related to **H** is

$$\boxed{W_H = \frac{1}{2}\int_{\text{vol}} \mathbf{B}\cdot\mathbf{H}\,dv} \tag{46}$$

Letting $\mathbf{B} = \mu\mathbf{H}$, we have the equivalent formulations

$$W_H = \frac{1}{2}\int_{\text{vol}} \mu H^2 dv \tag{47}$$

or

$$W_H = \frac{1}{2}\int_{\text{vol}} \frac{B^2}{\mu} dv \tag{48}$$

It is again convenient to think of this energy as being distributed throughout the volume with an energy density of $\frac{1}{2}\mathbf{B}\cdot\mathbf{H}$ J/m^3, although we have no mathematical justification for such a statement.

In spite of the fact that these results are valid only for linear media, we may use them to calculate the forces on nonlinear magnetic materials if we focus our attention on the linear media (usually air) which may surround them. For example, suppose that we have a long solenoid with a silicon-steel core. A coil containing n turns/m with a current I surrounds it. The magnetic field intensity in the core is therefore nI A $\cdot$ t/m, and the magnetic flux density can be obtained from the magnetization curve for silicon steel. Let us call this value B_{st}. Suppose that the core is composed of two semi-infinite cylinders[2] that are just touching. We now apply a mechanical force to separate these two sections of the core while keeping the flux density constant. We apply a force F over a distance dL, thus doing work $F\,dL$. Faraday's law does not

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[2] A semi-infinite cylinder is a cylinder of infinite length having one end located in finite space.

apply here, for the fields in the core have not changed, and we can therefore use the principle of virtual work to determine that the work we have done in moving one core appears as stored energy in the air gap we have created. By (48), this increase is

$$dW_H = F\,dL = \frac{1}{2}\frac{B_{st}^2}{\mu_0} S\,dL$$

where S is the core cross-sectional area. Thus

$$F = \frac{B_{st}^2 S}{2\mu_0}$$

If, for example, the magnetic field intensity is sufficient to produce saturation in the steel, approximately 1.4 T, the force is

$$F = 7.80 \times 10^5 S \quad \text{N}$$

or about 113 lb_f/in^2.

D8.11. (*a*) What force is being exerted on the pole faces of the circuit described in Problem D8.9 and Figure 8.13? (*b*) Is the force trying to open or close the air gap?

Ans. 1194 N; as Wilhelm Eduard Weber would put it, “schliessen”

8.10 INDUCTANCE AND MUTUAL INDUCTANCE

Inductance is the last of the three familiar parameters from circuit theory that we are defining in more general terms. Resistance was defined in Chapter 5 as the ratio of the potential difference between two equipotential surfaces of a conducting material to the total current crossing either equipotential surface. The resistance is a function of conductor geometry and conductivity only. Capacitance was defined in the same chapter as the ratio of the total charge on either of two equipotential conducting surfaces to the potential difference between the surfaces. Capacitance is a function only of the geometry of the two conducting surfaces and the permittivity of the dielectric medium between or surrounding them.

As a prelude to defining inductance, we first need to introduce the concept of flux linkage. Let us consider a toroid of N turns in which a current I produces a total flux Φ. We assume first that this flux links or encircles each of the N turns, and we also see that each of the N turns links the total flux Φ. The *flux linkage* $N\Phi$ is defined as the product of the number of turns N and the flux Φ linking each of them.[3] For a coil having a single turn, the flux linkage is equal to the total flux.

WWW Illustrations

[3] The symbol λ is commonly used for flux linkages. We will only occasionally use this concept, however, and we will continue to write it as $N\Phi$.

We now define *inductance* (or self-inductance) as the ratio of the total flux linkages to the current which they link,

$$\boxed{L = \frac{N\Phi}{I}} \tag{49}$$

The current I flowing in the N-turn coil produces the total flux Φ and $N\Phi$ flux linkages, where we assume for the moment that the flux Φ links each turn. This definition is applicable only to magnetic media which are linear, so that the flux is proportional to the current. If ferromagnetic materials are present, there is no single definition of inductance which is useful in all cases, and we shall restrict our attention to linear materials.

The unit of inductance is the henry (H), equivalent to one weber-turn per ampere.

Let us apply (49) in a straightforward way to calculate the inductance per meter length of a coaxial cable of inner radius a and outer radius b. We may take the expression for total flux developed as Eq. (42) in Chapter 7,

$$\Phi = \frac{\mu_0 Id}{2\pi} \ln \frac{b}{a}$$

and obtain the inductance rapidly for a length d,

$$L = \frac{\mu_0 d}{2\pi} \ln \frac{b}{a} \quad \text{H}$$

or, on a per-meter basis,

$$L = \frac{\mu_0}{2\pi} \ln \frac{b}{a} \quad \text{H/m} \tag{50}$$

In this case, $N = 1$ turn, and all the flux links all the current.

In the problem of a toroidal coil of N turns and a current I, as shown in Figure 7.12b, we have

$$B_\phi = \frac{\mu_0 NI}{2\pi\rho}$$

If the dimensions of the cross section are small compared with the mean radius of the toroid ρ_0, then the total flux is

$$\Phi = \frac{\mu_0 NIS}{2\pi\rho_0}$$

where S is the cross-sectional area. Multiplying the total flux by N, we have the flux linkages, and dividing by I, we have the inductance

$$L = \frac{\mu_0 N^2 S}{2\pi\rho_0} \tag{51}$$

Once again we have assumed that all the flux links all the turns, and this is a good assumption for a toroidal coil of many turns packed closely together. Suppose, however, that our toroid has an appreciable spacing between turns, a short part of which might look like Figure 8.14. The flux linkages are no longer the product of the

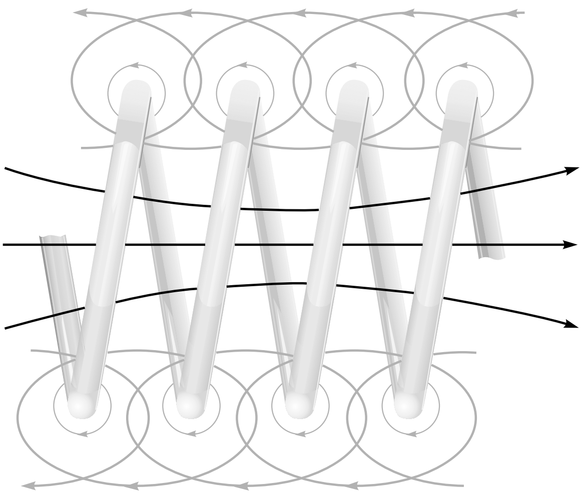

Figure 8.14 A portion of a coil showing partial flux linkages. The total flux linkages are obtained by adding the fluxes linking each turn.

flux at the mean radius times the total number of turns. In order to obtain the total flux linkages we must look at the coil on a turn-by-turn basis.

$$(N\Phi)_{\text{total}} = \Phi_1 + \Phi_2 + \cdots + \Phi_i + \cdots + \Phi_N$$
$$= \sum_{i=1}^{N} \Phi_i$$

where Φ_i is the flux linking the ith turn. Rather than doing this, we usually rely on experience and empirical quantities called winding factors and pitch factors to adjust the basic formula to apply to the real physical world.

An equivalent definition for inductance may be made using an energy point of view,

$$\boxed{L = \frac{2W_H}{I^2}} \tag{52}$$

where I is the total current flowing in the closed path and W_H is the energy in the magnetic field produced by the current. After using (52) to obtain several other general expressions for inductance, we will show that it is equivalent to (49). We first express the potential energy W_H in terms of the magnetic fields,

$$L = \frac{\int_{\text{vol}} \mathbf{B} \cdot \mathbf{H}\, dv}{I^2} \tag{53}$$

and then replace $\mathbf{B}$ by $\nabla \times \mathbf{A}$,

$$L = \frac{1}{I^2}\int_{\text{vol}} \mathbf{H} \cdot (\nabla \times \mathbf{A})\, dv$$

The vector identity

$$\nabla \cdot (\mathbf{A} \times \mathbf{H}) \equiv \mathbf{H} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{H}) \tag{54}$$

may be proved by expansion in rectangular coordinates. The inductance is then

$$L = \frac{1}{I^2}\left[\int_{\text{vol}} \nabla \cdot (\mathbf{A} \times \mathbf{H})\, dv + \int_{\text{vol}} \mathbf{A} \cdot (\nabla \times \mathbf{H})\, dv\right] \tag{55}$$

After applying the divergence theorem to the first integral and letting $\nabla \times \mathbf{H} = \mathbf{J}$ in the second integral, we have

$$L = \frac{1}{I^2}\left[\oint_S (\mathbf{A} \times \mathbf{H}) \cdot d\mathbf{S} + \int_{\text{vol}} \mathbf{A} \cdot \mathbf{J}\, dv\right]$$

The surface integral is zero, as the surface encloses the volume containing all the magnetic energy, and this requires that **A** and **H** be zero on the bounding surface. The inductance may therefore be written as

$$L = \frac{1}{I^2}\int_{\text{vol}} \mathbf{A} \cdot \mathbf{J}\, dv \tag{56}$$

Equation (56) expresses the inductance in terms of an integral of the values of **A** and **J** at every point. Because current density exists only within the conductor, the integrand is zero at all points *outside* the conductor, and the vector magnetic potential need not be determined there. The vector potential is that which arises from the current **J**, and any other current source contributing a vector potential field in the region of the original current density is to be ignored for the present. Later we will see that this leads to a *mutual inductance*.

The vector magnetic potential **A** due to **J** is given by Eq. (51), Chapter 7,

$$\mathbf{A} = \int_{\text{vol}} \frac{\mu \mathbf{J}}{4\pi R} dv$$

and the inductance may therefore be expressed more basically as a rather formidable double volume integral,

$$L = \frac{1}{I^2}\int_{\text{vol}} \left(\int_{\text{vol}} \frac{\mu \mathbf{J}}{4\pi R} dv\right) \cdot \mathbf{J}\, dv \tag{57}$$

A slightly simpler integral expression is obtained by restricting our attention to current filaments of small cross section for which $\mathbf{J}\, dv$ may be replaced by $I\, d\mathbf{L}$ and the volume integral by a closed line integral along the axis of the filament,

$$\begin{aligned} L &= \frac{1}{I^2}\oint \left(\oint \frac{\mu I\, d\mathbf{L}}{4\pi R}\right) \cdot I\, d\mathbf{L} \\ &= \frac{\mu}{4\pi}\oint \left(\oint \frac{d\mathbf{L}}{R}\right) \cdot d\mathbf{L} \end{aligned} \tag{58}$$

Our only present interest in Eqs. (57) and (58) lies in their implication that the inductance is a function of the distribution of the current in space or the geometry of the conductor configuration.

To obtain our original definition of inductance (49), let us hypothesize a uniform current distribution in a filamentary conductor of small cross section so that $\mathbf{J}\, dv$

in (56) becomes $I\,d\mathbf{L}$,

$$L = \frac{1}{I}\oint \mathbf{A}\cdot d\mathbf{L} \tag{59}$$

For a small cross section, $d\mathbf{L}$ may be taken along the center of the filament. We now apply Stokes' theorem and obtain

$$L = \frac{1}{I}\int_S (\nabla \times \mathbf{A})\cdot d\mathbf{S}$$

or

$$L = \frac{1}{I}\int_S \mathbf{B}\cdot d\mathbf{S}$$

or

$$L = \frac{\Phi}{I} \tag{60}$$

Retracing the steps by which (60) is obtained, we should see that the flux Φ is that portion of the total flux that passes through any and every open surface whose perimeter is the filamentary current path.

If we now let the filament make N identical turns about the total flux, an idealization that may be closely realized in some types of inductors, the closed line integral must consist of N laps about this common path, and (60) becomes

$$L = \frac{N\Phi}{I} \tag{61}$$

The flux Φ is now the flux crossing any surface whose perimeter is the path occupied by any *one* of the N turns. The inductance of an N-turn coil may still be obtained from (60), however, if we realize that the flux is that which crosses the complicated surface[4] whose perimeter consists of all N turns.

Use of any of the inductance expressions for a true filamentary conductor (having zero radius) leads to an infinite value of inductance, regardless of the configuration of the filament. Near the conductor, Ampère's circuital law shows that the magnetic field intensity varies inversely with the distance from the conductor, and a simple integration soon shows that an infinite amount of energy and an infinite amount of flux are contained within any finite cylinder about the filament. This difficulty is eliminated by specifying a small but finite filamentary radius.

The interior of any conductor also contains magnetic flux, and this flux links a variable fraction of the total current, depending on its location. These flux linkages lead to an *internal inductance,* which must be combined with the external inductance to obtain the total inductance. The internal inductance of a long, straight wire of circular cross section, radius a, and uniform current distribution is

$$\boxed{L_{a,\mathrm{int}} = \frac{\mu}{8\pi} \quad \mathrm{H/m}} \tag{62}$$

a result requested in Problem 8.43 at the end of this chapter.

[4] Somewhat like a spiral ramp.

In Chapter 11, we will see that the current distribution in a conductor at high frequencies tends to be concentrated near the surface. The internal flux is reduced, and it is usually sufficient to consider only the external inductance. At lower frequencies, however, internal inductance may become an appreciable part of the total inductance.

We conclude by defining the *mutual inductance* between circuits 1 and 2, M_{12}, in terms of mutual flux linkages,

$$\boxed{M_{12} = \frac{N_2 \Phi_{12}}{I_1}} \tag{63}$$

where Φ_{12} signifies the flux produced by I_1 which links the path of the filamentary current I_2, and N_2 is the number of turns in circuit 2. The mutual inductance, therefore, depends on the magnetic interaction between two currents. With either current alone, the total energy stored in the magnetic field can be found in terms of a single inductance, or self-inductance; with both currents having nonzero values, the total energy is a function of the two self-inductances and the mutual inductance. In terms of a mutual energy, it can be shown that (63) is equivalent to

$$M_{12} = \frac{1}{I_1 I_2} \int_{\text{vol}} (\mathbf{B}_1 \cdot \mathbf{H}_2) d\nu \tag{64}$$

or

$$M_{12} = \frac{1}{I_1 I_2} \int_{\text{vol}} (\mu \mathbf{H}_1 \cdot \mathbf{H}_2) d\nu \tag{65}$$

where $\mathbf{B}_1$ is the field resulting from I_1 (with $I_2 = 0$) and $\mathbf{H}_2$ is the field arising from I_2 (with $I_1 = 0$). Interchange of the subscripts does not change the right-hand side of (65), and therefore

$$\boxed{M_{12} = M_{21}} \tag{66}$$

Mutual inductance is also measured in henrys, and we rely on the context to allow us to differentiate it from magnetization, also represented by M.

EXAMPLE 8.9

Calculate the self-inductances of and the mutual inductances between two coaxial solenoids of radius R_1 and R_2, $R_2 > R_1$, carrying currents I_1 and I_2 with n_1 and n_2 turns/m, respectively.

Solution. We first attack the mutual inductances. From Eq. (15), Chapter 7, we let $n_1 = N/d$, and obtain

$$\begin{aligned} \mathbf{H}_1 &= n_1 I_1 \mathbf{a}_z \quad (0 < \rho < R_1) \\ &= 0 \quad (\rho > R_1) \end{aligned}$$

and

$$\begin{aligned} \mathbf{H}_2 &= n_2 I_2 \mathbf{a}_z \quad (0 < \rho < R_2) \\ &= 0 \quad (\rho > R_2) \end{aligned}$$

Thus, for this uniform field

$$\Phi_{12} = \mu_0 n_1 I_1 \pi R_1^2$$

and

$$M_{12} = \mu_0 n_1 n_2 \pi R_1^2$$

Similarly,

$$\Phi_{21} = \mu_0 n_2 I_2 \pi R_1^2$$
$$M_{21} = \mu_0 n_1 n_2 \pi R_1^2 = M_{12}$$

If $n_1 = 50$ turns/cm, $n_2 = 80$ turns/cm, $R_1 = 2$ cm, and $R_2 = 3$ cm, then

$$M_{12} = M_{21} = 4\pi \times 10^{-7}(5000)(8000)\pi(0.02^2) = 63.2 \text{ mH/m}$$

The self-inductances are easily found. The flux produced in coil 1 by I_1 is

$$\Phi_{11} = \mu_0 n_1 I_1 \pi R_1^2$$

and thus

$$L_1 = \mu_0 n_1^2 S_1 d \quad \text{H}$$

The inductance per unit length is therefore

$$L_1 = \mu_0 n_1^2 S_1 \quad \text{H/m}$$

or

$$L_1 = 39.5 \quad \text{mH/m}$$

Similarly,

$$L_2 = \mu_0 n_2^2 S_2 = 22.7 \quad \text{mH/m}$$

We see, therefore, that there are many methods available for the calculation of self-inductance and mutual inductance. Unfortunately, even problems possessing a high degree of symmetry present very challenging integrals for evaluation, and only a few problems are available for us to try our skill on.

Inductance will be discussed in circuit terms in Chapter 10.

D8.12. Calculate the self-inductance of: (*a*) 3.5 m of coaxial cable with $a = 0.8$ mm and $b = 4$ mm, filled with a material for which $\mu_r = 50$; (*b*) a toroidal coil of 500 turns, wound on a fiberglass form having a 2.5×2.5 cm square cross section and an inner radius of 2 cm; (*c*) a solenoid having 500 turns about a cylindrical core of 2 cm radius in which $\mu_r = 50$ for $0 < \rho < 0.5$ cm and $\mu_r = 1$ for $0.5 < \rho < 2$ cm; the length of the solenoid is 50 cm.

Ans. 56.3 μH; 1.01 mH; 3.2 mH

D8.13. A solenoid is 50 cm long, 2 cm in diameter, and contains 1500 turns. The cylindrical core has a diameter of 2 cm and a relative permeability of 75. This coil is coaxial with a second solenoid, also 50 cm long, but with a 3 cm diameter and 1200 turns. Calculate: (*a*) L for the inner solenoid; (*b*) L for the outer solenoid; (*c*) M between the two solenoids.

Ans. 133.2 mH; 192 mH; 106.6 mH

REFERENCES

1. Kraus, J. D., and D. A. Fleisch. (See References for Chapter 3.) Examples of the calculation of inductance are given on pp. 99–108.
2. Matsch, L. W. (See References for Chapter 6.) Chapter 3 is devoted to magnetic circuits and ferromagnetic materials.
3. Paul, C. R., K. W. Whites, and S. Y. Nasar. (See References for Chapter 7.) Magnetic circuits, including those with permanent magnets, are discussed on pp. 263–70.

CHAPTER 8 PROBLEMS

WW Quizzes

8.1 A point charge, $Q = -0.3\ \mu\text{C}$ and $m = 3 \times 10^{-16}$ kg, is moving through the field $\mathbf{E} = 30\mathbf{a}_z$ V/m. Use Eq. (1) and Newton's laws to develop the appropriate differential equations and solve them, subject to the initial conditions at $t = 0$, $\mathbf{v} = 3 \times 10^5\mathbf{a}_x$ m/s at the origin. At $t = 3\ \mu$s, find (*a*) the position $P(x, y, z)$ of the charge; (*b*) the velocity $\mathbf{v}$; (*c*) the kinetic energy of the charge.

8.2 Compare the magnitudes of the electric and magnetic forces on an electron that has attained a velocity of 10^7 m/s. Assume an electric field intensity of 10^5 V/m, and a magnetic flux density associated with that of the Earth's magnetic field in temperate latitudes, 0.5 gauss.

8.3 A point charge for which $Q = 2 \times 10^{-16}$ C and $m = 5 \times 10^{-26}$ kg is moving in the combined fields $\mathbf{E} = 100\mathbf{a}_x - 200\mathbf{a}_y + 300\mathbf{a}_z$ V/m and $\mathbf{B} = -3\mathbf{a}_x + 2\mathbf{a}_y - \mathbf{a}_z$ mT. If the charge velocity at $t = 0$ is $\mathbf{v}(0) = (2\mathbf{a}_x - 3\mathbf{a}_y - 4\mathbf{a}_z)10^5$ m/s (*a*) give the unit vector showing the direction in which the charge is accelerating at $t = 0$; (*b*) find the kinetic energy of the charge at $t = 0$.

8.4 Show that a charged particle in a uniform magnetic field describes a circular orbit with an orbital period that is independent of the radius. Find the relationship between the angular velocity and magnetic flux density for an electron (the *cyclotron frequency*).

8.5 A rectangular loop of wire in free space joins point $A(1, 0, 1)$ to point $B(3, 0, 1)$ to point $C(3, 0, 4)$ to point $D(1, 0, 4)$ to point A. The wire carries a current of 6 mA, flowing in the $\mathbf{a}_z$ direction from B to C. A filamentary current of 15 A flows along the entire z axis in the $\mathbf{a}_z$ direction. (a) Find $\mathbf{F}$ on side BC. (b) Find $\mathbf{F}$ on side AB. (c) Find $\mathbf{F}_{\text{total}}$ on the loop.

8.6 Show that the differential work in moving a current element $Id\mathbf{L}$ through a distance $d\mathbf{l}$ in a magetic field $\mathbf{B}$ is the negative of that done in moving the element $Id\mathbf{l}$ through a distance $d\mathbf{L}$ in the same field.

8.7 Uniform current sheets are located in free space as follows: $8\mathbf{a}_z$ A/m at $y = 0$, $-4\mathbf{a}_z$ A/m at $y = 1$, and $-4\mathbf{a}_z$ A/m at $y = -1$. Find the vector force per meter length exerted on a current filament carrying 7 mA in the $\mathbf{a}_L$ direction if the filament is located at (a) $x = 0$, $y = 0.5$, and $\mathbf{a}_L = \mathbf{a}_z$; ($b$) $y = 0.5$, $z = 0$, and $\mathbf{a}_L = \mathbf{a}_x$; ($c$) $x = 0$, $y = 1.5$, and $\mathbf{a}_L = \mathbf{a}_z$.

8.8 Two conducting strips, having infinite length in the z direction, lie in the xz plane. One occupies the region $d/2 < x < b + d/2$ and carries surface current density $\mathbf{K} = K_0\mathbf{a}_z$; the other is situated at $-(b + d/2) < x < -d/2$ and carries surface current density $-K_0\mathbf{a}_z$. (a) Find the force per unit length in z that tends to separate the two strips. (b) Let b approach zero while maintaining constant current, $I = K_0 b$, and show that the force per unit length approaches $\mu_0 I^2/(2\pi d)$ N/m.

8.9 A current of $-100\mathbf{a}_z$ A/m flows on the conducting cylinder $\rho = 5$ mm, and $+500\mathbf{a}_z$ A/m is present on the conducting cylinder $\rho = 1$ mm. Find the magnitude of the total force per meter length that is acting to split the outer cylinder apart along its length.

8.10 A planar transmission line consists of two conducting planes of width b separated d m in air, carrying equal and opposite currents of I A. If $b \gg d$, find the force of repulsion per meter of length between the two conductors.

8.11 (a) Use Eq. (14), Section 8.3, to show that the force of attraction per unit length between two filamentary conductors in free space with currents $I_1\mathbf{a}_z$ at $x = 0$, $y = d/2$, and $I_2\mathbf{a}_z$ at $x = 0$, $y = -d/2$, is $\mu_0 I_1 I_2/(2\pi d)$. (b) Show how a simpler method can be used to check your result.

8.12 Two circular wire rings are parallel to each other, share the same axis, are of radius a, and are separated by distance d, where $d << a$. Each ring carries current I. Find the approximate force of attraction and indicate the relative orientations of the currents.

8.13 A current of 6 A flows from $M(2, 0, 5)$ to $N(5, 0, 5)$ in a straight, solid conductor in free space. An infinite current filament lies along the z axis and carries 50 A in the $\mathbf{a}_z$ direction. Compute the vector torque on the wire segment using an origin at: (a) (0, 0, 5); (b) (0, 0, 0); (c) (3, 0, 0).

8.14 A solenoid is 25 cm long, 3 cm in diameter, and carries 4 A dc in its 400 turns. Its axis is perpendicular to a uniform magnetic field of 0.8 Wb/m^2 in air. Using an origin at the center of the solenoid, calculate the torque acting on it.

8.15 A solid conducting filament extends from $x = -b$ to $x = b$ along the line $y = 2$, $z = 0$. This filament carries a current of 3 A in the $\mathbf{a}_x$ direction. An infinite filament on the z axis carries 5 A in the $\mathbf{a}_z$ direction. Obtain an expression for the torque exerted on the finite conductor about an origin located at (0, 2, 0).

8.16 Assume that an electron is describing a circular orbit of radius a about a positively charged nucleus. (a) By selecting an appropriate current and area, show that the equivalent orbital dipole moment is $ea^2\omega/2$, where ω is the electron's angular velocity. (b) Show that the torque produced by a magnetic field parallel to the plane of the orbit is $ea^2\omega B/2$. (c) By equating the Coulomb and centrifugal forces, show that ω is $(4\pi\epsilon_0 m_e a^3/e^2)^{-1/2}$, where m_e is the electron mass. (d) Find values for the angular velocity, torque, and the orbital magnetic moment for a hydrogen atom, where a is about 6×10^{-11} m; let $B = 0.5$ T.

8.17 The hydrogen atom described in Problem 8.16 is now subjected to a magnetic field having the same direction as that of the atom. Show that the forces caused by B result in a decrease of the angular velocity by $eB/(2m_e)$ and a decrease in the orbital moment by $e^2a^2B/(4m_e)$. What are these decreases for the hydrogen atom in parts per million for an external magnetic flux density of 0.5 T?

8.18 Calculate the vector torque on the square loop shown in Figure 8.15 about an origin at A in the field B, given (a) $A(0, 0, 0)$ and $\mathbf{B} = 100\mathbf{a}_y$ mT; (b) $A(0, 0, 0)$ and $\mathbf{B} = 200\mathbf{a}_x + 100\mathbf{a}_y$ mT; (c) $A(1, 2, 3)$ and $\mathbf{B} = 200\mathbf{a}_x + 100\mathbf{a}_y - 300\mathbf{a}_z$ mT; (d) $A(1, 2, 3)$ and $\mathbf{B} = 200\mathbf{a}_x + 100\mathbf{a}_y - 300\mathbf{a}_z$ mT for $x \geq 2$ and $\mathbf{B} = 0$ elsewhere.

8.19 Given a material for which $\chi_m = 3.1$ and within which $\mathbf{B} = 0.4y\mathbf{a}_z$ T, find (a)$\mathbf{H}$; (b) μ; (c) μ_r; (d) $\mathbf{M}$; (e) $\mathbf{J}$; (f) $\mathbf{J}_B$; (g) $\mathbf{J}_T$.

8.20 Find $\mathbf{H}$ in a material where (a) $\mu_r = 4.2$, there are 2.7×10^{29} atoms/m^3, and each atom has a dipole moment of $2.6 \times 10^{-30}\mathbf{a}_y$ A $\cdot$ m^2; (b) $\mathbf{M} = 270\mathbf{a}_z$ A/m and $\mu = 2\mu$ H/m; (c) $\chi_m = 0.7$ and $\mathbf{B} = 2\mathbf{a}_z$ T. (d) Find $\mathbf{M}$ in a material where bound surface current densities of $12\mathbf{a}_z$ A/m and $-9\mathbf{a}_z$ A/m exist at $\rho = 0.3$ m and 0.4 m, respectively.

8.21 Find the magnitude of the magnetization in a material for which (a) the magnetic flux density is 0.02 Wb/m^2; (b) the magnetic field intensity is 1200 A/m and the relative permeability is 1.005; (c) there are 7.2×10^{28} atoms per cubic meter, each having a dipole moment of 4×10^{-30} A·m^2 in the same direction, and the magnetic susceptibility is 0.003.

8.22 Under some conditions, it is possible to approximate the effects of ferromagnetic materials by assuming linearity in the relationship of $\mathbf{B}$ and $\mathbf{H}$. Let $\mu_r = 1000$ for a certain material of which a cylindrical wire of radius 1 mm is made. If $I = 1$ A and the current distribution is uniform, find (a) $\mathbf{B}$, (b) $\mathbf{H}$, (c) $\mathbf{M}$, (d) $\mathbf{J}$, and (e) $\mathbf{J}_B$ within the wire.

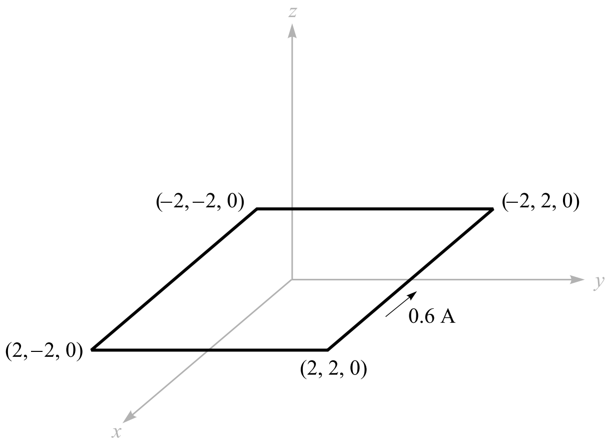

Figure 8.15 See Problem 8.18.

8.23 Calculate values for H_ϕ, B_ϕ, and M_ϕ at $\rho = c$ for a coaxial cable with $a = 2.5$ mm and $b = 6$ mm if it carries a current $I = 12$ A in the center conductor, and $\mu = 3\mu$H/m for $2.5\,\text{mm} < \rho < 3.5\,\text{mm}$, $\mu = 5\,\mu$H/m for $3.5\,\text{mm} < \rho < 4.5\,\text{mm}$, and $\mu = 10\,\mu$H/m for $4.5\,\text{mm} < \rho < 6$ mm. Use $c =$: (a) 3 mm; (b) 4 mm; (c) 5 mm.

8.24 Two current sheets, $K_0\mathbf{a}_y$ A/m at $z = 0$ and $-K_0\mathbf{a}_y$ A/m at $z = d$, are separated by an inhomogeneous material for which $\mu_r = az + 1$, where a is a constant. (a) Find expressions for **H** and **B** in the material. (b) Find the total flux that crosses a 1m^2 area on the yz plane.

8.25 A conducting filament at $z = 0$ carries 12 A in the $\mathbf{a}_z$ direction. Let $\mu_r = 1$ for $\rho < 1$ cm, $\mu_r = 6$ for $1 < \rho < 2$ cm, and $\mu_r = 1$ for $\rho > 2$ cm. Find: (a) **H** everywhere; (b) **B** everywhere.

8.26 A long solenoid has a radius of 3 cm, 5000 turns/m, and carries current $I = 0.25$ A. The region $0 < \rho < a$ within the solenoid has $\mu_r = 5$, whereas $\mu_r = 1$ for $a < \rho < 3$ cm. Determine a so that (a) a total flux of 10 μWb is present; (b) the flux is equally divided between the regions $0 < \rho < a$ and $a < \rho < 3$ cm.

8.27 Let $\mu_{r1} = 2$ in region 1, defined by $2x + 3y - 4z > 1$, while $\mu_{r2} = 5$ in region 2 where $2x + 3y - 4z < 1$. In region 1, $\mathbf{H}_1 = 50\mathbf{a}_x - 30\mathbf{a}_y + 20\mathbf{a}_z$ A/m. Find (a) $\mathbf{H}_{N1}$; (b) $\mathbf{H}_{t1}$; (c) $\mathbf{H}_{t2}$; (d) $\mathbf{H}_{N2}$; (e) θ_1, the angle between $\mathbf{H}_1$ and $\mathbf{a}_{N21}$; (f) θ_2, the angle between $\mathbf{H}_2$ and $\mathbf{a}_{N21}$.

8.28 For values of B below the knee on the magnetization curve for silicon steel, approximate the curve by a straight line with $\mu = 5$ mH/m. The core shown in Figure 8.16 has areas of 1.6 cm^2 and lengths of 10 cm in each outer leg, and an area of 2.5 cm^2 and a length of 3 cm in the central leg. A coil of 1200 turns carrying 12 mA is placed around the central leg. Find B in the (a) center leg; (b) center leg if a 0.3 mm air gap is present in the center leg.

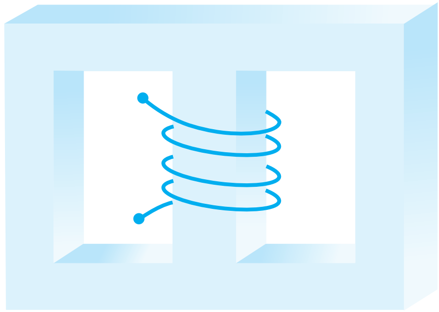

Figure 8.16 See Problem 8.28.

8.29 In Problem 8.28, the linear approximation suggested in the statement of the problem leads to flux density of 0.666 T in the central leg. Using this value of B and the magnetization curve for silicon steel, what current is required in the 1200-turn coil?

8.30 A rectangular core has fixed permeability $\mu_r >> 1$, a square cross section of dimensions $a \times a$, and has centerline dimensions around its perimeter of b and d. Coils 1 and 2, having turn numbers N_1 and N_2, are wound on the core. Consider a selected core cross-sectional plane as lying within the xy plane, such that the surface is defined by $0 < x < a$, $0 < y < a$. (a) With current I_1 in coil 1, use Ampere's circuital law to find the magnetic flux density as a function of position over the core cross-section. (b) Integrate your result of part (a) to determine the total magnetic flux within the core. (c) Find the self-inductance of coil 1. (d) Find the mutual inductance between coils 1 and 2.

8.31 A toroid is constructed of a magnetic material having a cross-sectional area of 2.5 cm^2 and an effective length of 8 cm. There is also a short air gap of 0.25 mm length and an effective area of 2.8 cm^2. An mmf of 200 $\text{A} \cdot \text{t}$ is applied to the magnetic circuit. Calculate the total flux in the toroid if the magnetic material: (a) is assumed to have infinite permeability; (b) is assumed to be linear with $\mu_r = 1000$; (c) is silicon steel.

8.32 (a) Find an expression for the magnetic energy stored per unit length in a coaxial transmission line consisting of conducting sleeves of negligible thickness, having radii a and b. A medium of relative permeability μ_r fills the region between conductors. Assume current I flows in both conductors in opposite directions. (b) Obtain the inductance, L, per unit length of line by equating the energy to $(1/2)LI^2$.

8.33 A toroidal core has a square cross section, $2.5\,\text{cm} < \rho < 3.5\,\text{cm}$, $-0.5\,\text{cm} < z < 0.5\,\text{cm}$. The upper half of the toroid, $0 < z < 0.5\,\text{cm}$, is constructed of a linear material for which $\mu_r = 10$, while the lower half, $-0.5\,\text{cm} < z < 0$,

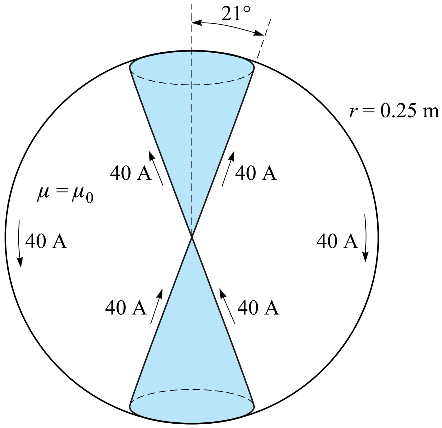

Figure 8.17 See Problem 8.35.

has $\mu_r = 20$. An mmf of 150 A · t establishes a flux in the $\mathbf{a}_\phi$ direction. For $z > 0$, find: (*a*) $H_\phi(\rho)$; (*b*) $B_\phi(\rho)$; (*c*) $\Phi_{z>0}$. (*d*) Repeat for $z > 0$. (*e*) Find Φ_{total}.

8.34 Determine the energy stored per unit length in the internal magnetic field of an infinitely long, straight wire of radius a, carrying uniform current I.

8.35 The cones $\theta = 21^\circ$ and $\theta = 159^\circ$ are conducting surfaces and carry total currents of 40 A, as shown in Figure 8.17. The currents return on a spherical conducting surface of 0.25 m radius. (*a*) Find **H** in the region $0 < r < 0.25$, $21^\circ < \theta < 159^\circ$, $0 < \phi < 2\pi$. (*b*) How much energy is stored in this region?

8.36 The dimensions of the outer conductor of a coaxial cable are b and c, where $c > b$. Assuming $\mu = \mu_0$, find the magnetic energy stored per unit length in the region $b < \rho < c$ for a uniformly distributed total current I flowing in opposite directions in the inner and outer conductors.

8.37 Find the inductance of the cone-sphere configuration described in Problem 8.35 and Figure 8.17. The inductance is that offered at the origin between the vertices of the cone.

8.38 A toroidal core has a rectangular cross section defined by the surfaces $\rho = 2$ cm, $\rho = 3$ cm, $z = 4$ cm, and $z = 4.5$ cm. The core material has a relative permeability of 80. If the core is wound with a coil containing 8000 turns of wire, find its inductance.

8.39 Conducting planes in air at $z = 0$ and $z = d$ carry surface currents of $\pm K_0\mathbf{a}_x$ A/m. (*a*) Find the energy stored in the magnetic field per unit length $(0 < x < 1)$ in a width $w(0 < y < w)$. (*b*) Calculate the inductance per unit length of this transmission line from $W_H = \frac{1}{2}LI^2$, where I is the total current in a width w in either conductor. (*c*) Calculate the total flux passing through the rectangle $0 < x < 1$, $0 < z < d$, in the plane $y = 0$, and from this result again find the inductance per unit length.

8.40 A coaxial cable has conductor radii a and b, where $a < b$. Material of permeability $\mu_r \neq 1$ exists in the region $a < \rho < c$, whereas the region $c < \rho < b$ is air filled. Find an expression for the inductance per unit length.

8.41 A rectangular coil is composed of 150 turns of a filamentary conductor. Find the mutual inductance in free space between this coil and an infinite straight filament on the z axis if the four corners of the coil are located at: (*a*) (0, 1, 0), (0, 3, 0), (0, 3, 1), and (0, 1, 1); (*b*) (1, 1, 0), (1, 3, 0), (1, 3, 1), and (1, 1, 1).

8.42 Find the mutual inductance between two filaments forming circular rings of radii a and Δa, where $\Delta a \ll a$. The field should be determined by approximate methods. The rings are coplanar and concentric.

8.43 (*a*) Use energy relationships to show that the internal inductance of a nonmagnetic cylindrical wire of radius a carrying a uniformly distributed current I is $\mu_0/(8\pi)$ H/m. (*b*) Find the internal inductance if the portion of the conductor for which $\rho < c < a$ is removed.

8.44 Show that the external inductance per unit length of a two-wire transmission line carrying equal and opposite currents is approximately $(\mu/\pi)\ln(d/a)$ H/m, where a is the radius of each wire and d is the center-to-center wire spacing. On what basis is the approximation valid?